

Os potenciais de uma carga pontual em movimento no MKS

Vou introduzir os cálculos dos potenciais de Liénard & Wiechert, escalar e vetorial, de uma partícula carregada executando um movimento com trajetória dada. Como queremos os campos causados pela partícula, utilizamos as soluções retardadas dos potenciais no calibre de Lorentz:

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{V_\infty} d^3r' \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}$$

e

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{V_\infty} d^3r' \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

Para uma carga q descrevendo uma trajetória $\mathbf{r}_0(t)$, temos

$$\rho(\mathbf{r}', t') = q\delta^{(3)}(\mathbf{r}' - \mathbf{r}_0(t'))$$

e

$$\mathbf{J}(\mathbf{r}', t') = q\mathbf{v}(t')\delta^{(3)}(\mathbf{r}' - \mathbf{r}_0(t')),$$

para quaisquer \mathbf{r}' e t' , onde

$$\mathbf{v}(t') = \frac{d\mathbf{r}_0(t')}{dt'}.$$

Assim, o potencial escalar, por exemplo, fica

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{V_\infty} d^3r' \frac{q\delta^{(3)}\left(\mathbf{r}' - \mathbf{r}_0\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

O problema é integrarmos

$$\int_{V_\infty} d^3r' \frac{\delta^{(3)}\left(\mathbf{r}' - \mathbf{r}_0\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

Para tal proeza, há um truque: é óbvio que a integral acima pode ser escrita como

$$\int_{-\infty}^{+\infty} dt' \int_{V_\infty} d^3r' \frac{\delta^{(3)}(\mathbf{r}' - \mathbf{r}_0(t'))}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t' - t + \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

e, portanto, integrando sobre a variável \mathbf{r}' , obtemos

$$\int_{V_\infty} d^3r' \frac{\delta^{(3)}\left(\mathbf{r}' - \mathbf{r}_0\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{|\mathbf{r}-\mathbf{r}'|} = \int_{-\infty}^{+\infty} dt' \frac{\delta\left(t' - t + \frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0(t')|}.$$

A seguir tomamos como fixos \mathbf{r} e t e introduzimos a função

$$f(t') = t' - t + \frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}.$$

Portanto, temos a seguinte integral para calcular:

$$\int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r}-\mathbf{r}_0(t')|}.$$

Das propriedades da função delta de Dirac, podemos escrever

$$\int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r}-\mathbf{r}_0(t')|} = \sum_k \int_{-\infty}^{+\infty} \frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|} \frac{\delta(t' - t_k)}{\left|\frac{df(t')}{dt'}\right|_{t'=t_k}},$$

onde os t_k 's são os instantes de tempo em que $f(t_k) = 0$. Em outras palavras, queremos os instantes de tempo t_k 's tais que

$$t_k = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_k)|}{c}.$$

O fato é que só há um instante de tempo para essa relação valer. Para vermos isso, suponhamos que existam dois instantes, t_1 e t_2 , com $t_1 \neq t_2$, tais que

$$t_1 = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_1)|}{c}$$

e

$$t_2 = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_2)|}{c}.$$

Sem perda de generalidade, suponhamos que $t_2 > t_1$. Logo,

$$c(t_2 - t_1) = |\mathbf{r} - \mathbf{r}_0(t_1)| - |\mathbf{r} - \mathbf{r}_0(t_2)|.$$

Mas,

$$\begin{aligned} |\mathbf{r} - \mathbf{r}_0(t_1)| - |\mathbf{r} - \mathbf{r}_0(t_2)| &\leq |\mathbf{r} - \mathbf{r}_0(t_1) - (\mathbf{r} - \mathbf{r}_0(t_2))| \\ &= |\mathbf{r}_0(t_2) - \mathbf{r}_0(t_1)|. \end{aligned}$$

Assim,

$$c \leq \frac{|\mathbf{r}_0(t_2) - \mathbf{r}_0(t_1)|}{t_2 - t_1},$$

implicando que a partícula deveria ir de $\mathbf{r}_0(t_1)$ até $\mathbf{r}_0(t_2)$ com uma velocidade média maior ou igual à velocidade da luz no vácuo. Como, por hipótese, estamos considerando uma partícula massiva, de massa $m > 0$, concluímos que há, no máximo, um instante de tempo em que a equação

$$t_k = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_k)|}{c}$$

vale e definimos esse instante como o tempo retardado:

$$t_R = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_R)|}{c}.$$

O resultado da integral acima pode ser escrito em termos do tempo retardado como

$$\begin{aligned} \int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r} - \mathbf{r}_0(t')|} &= \int_{-\infty}^{+\infty} \frac{dt'}{|\mathbf{r} - \mathbf{r}_0(t')|} \frac{\delta(t' - t_R)}{\left| \frac{df(t')}{dt'} \right|_{t'=t_R}} \\ &= \frac{1}{|\mathbf{r} - \mathbf{r}_0(t_R)| \left| \frac{df(t')}{dt'} \right|_{t'=t_R}}. \end{aligned}$$

Calculemos, explicitamente, a derivada temporal da função f :

$$\begin{aligned} \frac{df(t')}{dt'} &= \frac{d}{dt'} \left(t' - t + \frac{|\mathbf{r} - \mathbf{r}_0(t')|}{c} \right) \\ &= 1 + \frac{1}{c} \frac{d|\mathbf{r} - \mathbf{r}_0(t')|}{dt'}. \end{aligned}$$

Notemos que

$$\begin{aligned}
\frac{d|\mathbf{r} - \mathbf{r}_0(t')|}{dt'} &= \frac{1}{2|\mathbf{r} - \mathbf{r}_0(t')|} \frac{d|\mathbf{r} - \mathbf{r}_0(t')|^2}{dt'} \\
&= \frac{1}{2|\mathbf{r} - \mathbf{r}_0(t')|} \frac{d[\mathbf{r} - \mathbf{r}_0(t')] \cdot [\mathbf{r} - \mathbf{r}_0(t')]}{dt'} \\
&= \frac{[\mathbf{r} - \mathbf{r}_0(t')]}{|\mathbf{r} - \mathbf{r}_0(t')|} \cdot \frac{d[\mathbf{r} - \mathbf{r}_0(t')]}{dt'} \\
&= -\frac{[\mathbf{r} - \mathbf{r}_0(t')]}{|\mathbf{r} - \mathbf{r}_0(t')|} \cdot \frac{d\mathbf{r}_0(t')}{dt'} \\
&= -\frac{[\mathbf{r} - \mathbf{r}_0(t')]}{|\mathbf{r} - \mathbf{r}_0(t')|} \cdot \mathbf{v}(t').
\end{aligned}$$

Portanto,

$$\int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r} - \mathbf{r}_0(t')|} = \frac{1}{\left| |\mathbf{r} - \mathbf{r}_0(t_R)| - [\mathbf{r} - \mathbf{r}_0(t_R)] \cdot \frac{\mathbf{v}(t_R)}{c} \right|}.$$

Como $|\mathbf{v}/c| < 1$, temos

$$\int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r} - \mathbf{r}_0(t')|} = \frac{1}{|\mathbf{r} - \mathbf{r}_0(t_R)| - [\mathbf{r} - \mathbf{r}_0(t_R)] \cdot \frac{\mathbf{v}(t_R)}{c}}$$

e os potenciais podem ser escritos como

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}_0(t_R)| - [\mathbf{r} - \mathbf{r}_0(t_R)] \cdot \frac{\mathbf{v}(t_R)}{c}}$$

e, analogamente,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q\mathbf{v}(t_R)}{|\mathbf{r} - \mathbf{r}_0(t_R)| - [\mathbf{r} - \mathbf{r}_0(t_R)] \cdot \frac{\mathbf{v}(t_R)}{c}},$$

onde

$$t_R = t - \frac{|\mathbf{r} - \mathbf{r}_0(t_R)|}{c}.$$

Esses são os chamados potenciais de Liénard-Wiechert. Para simplificar a notação, definamos

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_0(t_R),$$

$$R = |\mathbf{R}|,$$

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R},$$

$$\begin{aligned}
\mathbf{v} &= \mathbf{v}(t_R) \\
&= \frac{d\mathbf{r}_0(t_R)}{dt_R} \\
&= \dot{\mathbf{r}}_0(t_R)
\end{aligned}$$

e

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}.$$

Com essas definições, podemos simplificar as expressões dos potenciais assim:

$$\phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R - \mathbf{R} \cdot \boldsymbol{\beta}},$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 c}{4\pi} \frac{q\boldsymbol{\beta}}{R - \mathbf{R} \cdot \boldsymbol{\beta}}$$

e

$$t_R = t - \frac{R}{c}.$$

Calculando o Campo Indução Magnética e o Campo Elétrico

Vamos agora calcular os campos \mathbf{B} e \mathbf{E} . Começemos com o cálculo de \mathbf{B} :

$$\mathbf{B} = \nabla \times \mathbf{A}$$

ou, em termos de componentes,

$$B_i = \sum_{j=1}^3 \sum_{k=1}^3 \varepsilon_{ijk} \partial_j A_k,$$

onde ε_{ijk} é o tensor de Levi-Civita, que é dado por

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{se } (i, j, k) \text{ for uma permutação par de } (1, 2, 3), \\ 0, & \text{se pelo menos dois dos índices } i, j, k \text{ forem iguais e} \\ -1, & \text{se } (i, j, k) \text{ for uma permutação ímpar de } (1, 2, 3). \end{cases}$$

Também utilizemos a notação

$$\partial_j = \frac{\partial}{\partial x_j},$$

para $j = 1, 2, 3$. A convenção de Einstein para somas permite que escrevamos

$$B_i = \varepsilon_{ijk} \partial_j A_k,$$

onde subentendemos que os índices j e k estão somados de 1 a 3, porque aparecem repetidos no mesmo termo. Temos, assim,

$$\begin{aligned} \partial_j A_k &= \frac{\mu_0}{4\pi} q \partial_j \left(\frac{v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \right) \\ &= \frac{\mu_0}{4\pi} q \left[\frac{\partial_j v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} + v_k \partial_j \left(\frac{1}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \right) \right] \\ &= \frac{\mu_0}{4\pi} q \left[\frac{\partial_j v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} - \frac{v_k}{(R - \mathbf{R} \cdot \boldsymbol{\beta})^2} (\partial_j R - \boldsymbol{\beta} \cdot \partial_j \mathbf{R} - \mathbf{R} \cdot \partial_j \boldsymbol{\beta}) \right]. \end{aligned}$$

Também,

$$\begin{aligned} \partial_j v_k &= \frac{dv_k}{dt_R} \partial_j t_R \\ &= a_k \partial_j t_R, \end{aligned}$$

Façamos agora o cálculo de $\partial_j t_R$:

$$\begin{aligned} \partial_j t_R &= \partial_j \left(t - \frac{R}{c} \right) \\ &= -\frac{1}{c} \partial_j R \\ &= -\frac{1}{c} \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j + \hat{\mathbf{R}} \cdot \boldsymbol{\beta} \partial_j t_R, \end{aligned}$$

ou seja,

$$(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}) \partial_j t_R = -\frac{1}{c} \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j,$$

resultando em

$$\partial_j t_R = -\frac{1}{c} \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})}.$$

Portanto,

$$\partial_j v_k = -\frac{a_k}{c} \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})},$$

$$\begin{aligned} \partial_j \mathbf{R} &= \partial_j \mathbf{r} - \partial_j \mathbf{r}_0(t_R) \\ &= \hat{\mathbf{x}}_j - \frac{d\mathbf{r}_0(t_R)}{dt_R} \partial_j t_R \\ &= \hat{\mathbf{x}}_j + \mathbf{v} \frac{1}{c} \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} \\ &= \hat{\mathbf{x}}_j + \boldsymbol{\beta} \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})}, \end{aligned}$$

$$\begin{aligned} \partial_j R &= \frac{1}{2R} \partial_j R^2 \\ &= \frac{1}{2R} \partial_j (\mathbf{R} \cdot \mathbf{R}) \\ &= \frac{\mathbf{R}}{R} \cdot \partial_j \mathbf{R} \\ &= \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j + \hat{\mathbf{R}} \cdot \boldsymbol{\beta} \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} \\ &= \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}) + \hat{\mathbf{R}} \cdot \boldsymbol{\beta} \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} \\ &= \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j - \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j \hat{\mathbf{R}} \cdot \boldsymbol{\beta} + \hat{\mathbf{R}} \cdot \boldsymbol{\beta} \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} \\ &= \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} \end{aligned}$$

e

$$\begin{aligned} \partial_j \boldsymbol{\beta} &= \frac{1}{c} \partial_j \mathbf{v} \\ &= \frac{1}{c} \frac{d\mathbf{v}}{dt_R} \partial_j t_R \\ &= -\frac{\mathbf{a}}{c^2} \frac{\hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})}, \end{aligned}$$

onde denotamos

$$a_k = \frac{dv_k}{dt_R}$$

e

$$\mathbf{a} = \frac{d\mathbf{v}}{dt_R}.$$

Assim,

$$\begin{aligned}
\partial_j A_k &= \frac{\mu_0}{4\pi} q \left[\frac{\partial_j v_k}{R - \mathbf{R} \cdot \boldsymbol{\beta}} - \frac{v_k}{(R - \mathbf{R} \cdot \boldsymbol{\beta})^2} (\partial_j R - \boldsymbol{\beta} \cdot \partial_j \mathbf{R} - \mathbf{R} \cdot \partial_j \boldsymbol{\beta}) \right] \\
&= \frac{\mu_0}{4\pi} q \left[\frac{\partial_j v_k}{R(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} - \frac{v_k (\partial_j R - \boldsymbol{\beta} \cdot \partial_j \mathbf{R} - R \hat{\mathbf{R}} \cdot \partial_j \boldsymbol{\beta})}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^2} \right] \\
&= \frac{\mu_0}{4\pi} q \left[\frac{\partial_j v_k}{R(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} + \frac{v_k \hat{\mathbf{R}} \cdot \partial_j \boldsymbol{\beta}}{R(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^2} - \frac{v_k (\partial_j R - \boldsymbol{\beta} \cdot \partial_j \mathbf{R})}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^2} \right].
\end{aligned}$$

Substituindo as derivadas parciais, temos

$$\begin{aligned}
\partial_j A_k &= \frac{\mu_0}{4\pi} q \left[\frac{-a_k \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{Rc(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^2} - \frac{v_k \hat{\mathbf{R}} \cdot \mathbf{a} \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{Rc^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right. \\
&\quad \left. - \frac{v_k}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^2} \left(\frac{(1 - \beta^2) \hat{\mathbf{R}} \cdot \hat{\mathbf{x}}_j}{(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})} - \beta_j \right) \right],
\end{aligned}$$

onde

$$\beta_j = \frac{v_j}{c}.$$

Em termos vetoriais, podemos escrever

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} q \left[\frac{-\hat{\mathbf{R}} \times \mathbf{a}}{Rc(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^2} - \frac{\hat{\mathbf{R}} \cdot \mathbf{a} \hat{\mathbf{R}} \times \mathbf{v}}{Rc^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right. \\
&\quad \left. - \frac{(1 - \beta^2) \hat{\mathbf{R}} \times \mathbf{v}}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right],
\end{aligned}$$

onde o termo proporcional a $\boldsymbol{\beta} \times \mathbf{v}$ se anula. Podemos ainda escrever

$$\begin{aligned}
\mathbf{B} &= \frac{\mu_0}{4\pi} q \hat{\mathbf{R}} \times \left[\frac{-\mathbf{a}(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}) - \hat{\mathbf{R}} \cdot \mathbf{a} \boldsymbol{\beta}}{Rc(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} - \frac{(1 - \beta^2) \mathbf{v}}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right] \\
&= \frac{\mu_0}{4\pi} q \hat{\mathbf{R}} \times \left[\frac{-\mathbf{a} \hat{\mathbf{R}} \cdot (\hat{\mathbf{R}} - \boldsymbol{\beta}) + \hat{\mathbf{R}} \cdot \mathbf{a} (\hat{\mathbf{R}} - \boldsymbol{\beta})}{Rc(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} - \frac{(1 - \beta^2) \mathbf{v}}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right],
\end{aligned}$$

onde adicionamos um termo proporcional a $\hat{\mathbf{R}}$ entre colchetes, que não contribui para \mathbf{B} , pois é multiplicado vetorialmente pelo $\hat{\mathbf{R}}$ que aparece fora dos colchetes. Notamos agora que

$$-\mathbf{a} \hat{\mathbf{R}} \cdot (\hat{\mathbf{R}} - \boldsymbol{\beta}) + \hat{\mathbf{R}} \cdot \mathbf{a} (\hat{\mathbf{R}} - \boldsymbol{\beta}) = \hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \mathbf{a}].$$

Logo,

$$\mathbf{B} = \frac{\mu_0}{4\pi} q \hat{\mathbf{R}} \times \left\{ \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \mathbf{a}]}{Rc(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} - \frac{(1 - \beta^2) \mathbf{v}}{R^2 (1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right\}.$$

De maneira análoga, calculamos o campo elétrico e obtemos

$$\begin{aligned}
 \mathbf{E} &= \frac{1}{4\pi\epsilon_0}q \left[\frac{(\hat{\mathbf{R}} - \boldsymbol{\beta})(1 - \beta^2)}{R^2(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} + \frac{(\hat{\mathbf{R}} - \boldsymbol{\beta})\hat{\mathbf{R}} \cdot \mathbf{a}}{Rc^2(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} - \frac{\mathbf{a}}{Rc^2(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^2} \right] \\
 &= \frac{1}{4\pi\epsilon_0}q \left[\frac{(\hat{\mathbf{R}} - \boldsymbol{\beta})(1 - \beta^2)}{R^2(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} + \frac{(\hat{\mathbf{R}} - \boldsymbol{\beta})\hat{\mathbf{R}} \cdot \mathbf{a} - \mathbf{a}(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})}{Rc^2(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right] \\
 &= \frac{1}{4\pi\epsilon_0}q \left[\frac{(\hat{\mathbf{R}} - \boldsymbol{\beta})(1 - \beta^2)}{R^2(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} + \frac{\hat{\mathbf{R}} \times [(\hat{\mathbf{R}} - \boldsymbol{\beta}) \times \mathbf{a}]}{Rc^2(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta})^3} \right]
 \end{aligned}$$

e, portanto,

$$\mathbf{B} = \frac{1}{c}\hat{\mathbf{R}} \times \mathbf{E}.$$