A densidade e o fluxo de energia no sistema MKS

Há textos de eletromagnetismo que utilizam o sistema de unidades CGS e há os que adotam o sistema MKS. Alguns fatores multiplicativos dependem do sistema de unidades. Já estabeleci as fórmulas práticas para o caso do sistema CGS na postagem, A densidade e o fluxo de energia. Na presente postagem vou repetir as considerações usando o sistema MKS de unidades.

Vamos utilizar a notação

$$\begin{aligned} \boldsymbol{\epsilon} &= \boldsymbol{\epsilon}_0 \exp\left(i\mathbf{k}\cdot\mathbf{r} - i\omega_k t\right), \\ \boldsymbol{\beta} &= \boldsymbol{\beta}_0 \exp\left(i\mathbf{k}\cdot\mathbf{r} - i\omega_k t\right), \end{aligned}$$

onde

$$\omega_k \equiv \frac{k}{\sqrt{\mu\varepsilon}}$$

e

$$\begin{aligned} \mathbf{E} &= \mathrm{Re}\left(\boldsymbol{\epsilon}\right), \\ \mathbf{B} &= \mathrm{Re}\left(\boldsymbol{\beta}\right). \end{aligned}$$

Assim, as equações de Maxwell dão

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{\epsilon} = 0 &\quad\Rightarrow\quad \mathbf{k}\cdot\boldsymbol{\epsilon}_0 = 0, \\ \boldsymbol{\nabla}\cdot\boldsymbol{\beta} = 0 &\quad\Rightarrow\quad \mathbf{k}\cdot\boldsymbol{\beta}_0 = 0, \\ \boldsymbol{\nabla}\times\boldsymbol{\epsilon} = -\frac{\partial\boldsymbol{\beta}}{\partial t} &\quad\Rightarrow\quad \mathbf{k}\times\boldsymbol{\epsilon}_0 = \omega_k\boldsymbol{\beta}_0, \\ \boldsymbol{\nabla}\times\boldsymbol{\beta} = \mu\varepsilon\frac{\partial\boldsymbol{\epsilon}}{\partial t} &\quad\Rightarrow\quad \mathbf{k}\times\boldsymbol{\beta}_0 = -\mu\varepsilon\omega_k\boldsymbol{\epsilon}_0. \end{aligned}$$

Em resumo,

$$\begin{aligned} \boldsymbol{\epsilon}_0 &= -\frac{1}{\sqrt{\mu\varepsilon}}\hat{\mathbf{k}}\times\boldsymbol{\beta}_0, \\ \boldsymbol{\beta}_0 &= \sqrt{\mu\varepsilon}\,\hat{\mathbf{k}}\times\boldsymbol{\epsilon}_0. \end{aligned}$$

A densidade de energia é dada por

$$\begin{aligned} u &= \frac{1}{2}\left(\mathbf{D}\cdot\mathbf{E} + \mathbf{H}\cdot\mathbf{B}\right) \\ &= \frac{\varepsilon}{2}\left[\mathrm{Re}\left(\boldsymbol{\epsilon}\right)\right]\cdot\left[\mathrm{Re}\left(\boldsymbol{\epsilon}\right)\right] + \frac{1}{2\mu}\left[\mathrm{Re}\left(\boldsymbol{\beta}\right)\right]\cdot\left[\mathrm{Re}\left(\boldsymbol{\beta}\right)\right]. \end{aligned}$$

O fluxo de energia é dado pelo vetor de Poynting:

$$\begin{aligned} \mathbf{S} &= \mathbf{E}\times\mathbf{H} \\ &= \frac{1}{\mu}\left[\mathrm{Re}\left(\boldsymbol{\epsilon}\right)\right]\times\left[\mathrm{Re}\left(\boldsymbol{\beta}\right)\right]. \end{aligned}$$

Se tomamos $\left[\mathrm{Re}\left(\boldsymbol{\epsilon}\right)\right]\times\left[\mathrm{Re}\left(\boldsymbol{\beta}\right)\right]$, por exemplo, obtemos o seguinte:

$$\begin{aligned} \left[\mathrm{Re}\left(\boldsymbol{\epsilon}\right)\right]\times\left[\mathrm{Re}\left(\boldsymbol{\beta}\right)\right] &= \frac{1}{4}\left(\boldsymbol{\epsilon}+\boldsymbol{\epsilon}^*\right)\times\left(\boldsymbol{\beta}+\boldsymbol{\beta}^*\right) \\ &= \frac{1}{4}\boldsymbol{\epsilon}\times\boldsymbol{\beta} + \frac{1}{4}\boldsymbol{\epsilon}\times\boldsymbol{\beta}^* + \frac{1}{4}\boldsymbol{\epsilon}^*\times\boldsymbol{\beta} + \frac{1}{4}\boldsymbol{\epsilon}^*\times\boldsymbol{\beta}^* \\ &= \frac{1}{2}\mathrm{Re}\left(\boldsymbol{\epsilon}\times\boldsymbol{\beta}^*\right) + \frac{1}{4}\boldsymbol{\epsilon}\times\boldsymbol{\beta} + \frac{1}{4}\boldsymbol{\epsilon}^*\times\boldsymbol{\beta}^*. \end{aligned}$$

Como $\mathbf{k}$ e ω_k são quantidades reais, então

$$\left(\boldsymbol{\epsilon}\times\boldsymbol{\beta}^*\right) = \left(\boldsymbol{\epsilon}_0\times\boldsymbol{\beta}_0^*\right)$$

não depende do tempo e não depende do espaço. A média temporal dos termos oscilantes restantes dá zero. Por exemplo,

$$\begin{aligned} \langle \boldsymbol{\epsilon} \times \boldsymbol{\beta} \rangle &\equiv (\boldsymbol{\epsilon}_0 \times \boldsymbol{\beta}_0) \exp(2i\mathbf{k}\cdot\mathbf{r}) \int_0^{\frac{2\pi}{\omega_k}} dt\, \exp(-2i\omega_k t) \\ &= \mathbf{0}. \end{aligned}$$

Assim,

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu} \mathrm{Re}\left(\boldsymbol{\epsilon} \times \boldsymbol{\beta}^*\right).$$

Procedendo similarmente para o caso da densidade de energia, obtemos a média temporal

$$\langle u \rangle = \frac{\varepsilon}{4} \boldsymbol{\epsilon} \cdot \boldsymbol{\epsilon}^* + \frac{1}{4\mu} \boldsymbol{\beta} \cdot \boldsymbol{\beta}^*.$$

Como $\mathbf{k}$ e ω_k são quantidades reais, então

$$\begin{aligned} \boldsymbol{\beta} \cdot \boldsymbol{\beta}^* &= \mu\varepsilon \left(\hat{\mathbf{k}} \times \boldsymbol{\epsilon}_0\right) \cdot \left(\hat{\mathbf{k}} \times \boldsymbol{\epsilon}_0^*\right) \\ &= \mu\varepsilon \hat{\mathbf{k}} \cdot \left[\boldsymbol{\epsilon}_0 \times \left(\hat{\mathbf{k}} \times \boldsymbol{\epsilon}_0^*\right)\right] \\ &= \mu\varepsilon \hat{\mathbf{k}} \cdot \left[\hat{\mathbf{k}} \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^* - \boldsymbol{\epsilon}_0^* \hat{\mathbf{k}} \cdot \boldsymbol{\epsilon}_0\right] \\ &= \mu\varepsilon \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^*, \\ \boldsymbol{\epsilon} \times \boldsymbol{\beta}^* &= \sqrt{\mu\varepsilon} \boldsymbol{\epsilon}_0 \times \left(\hat{\mathbf{k}} \times \boldsymbol{\epsilon}_0^*\right) \\ &= \sqrt{\mu\varepsilon} \left(\hat{\mathbf{k}} \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^* - \boldsymbol{\epsilon}_0^* \hat{\mathbf{k}} \cdot \boldsymbol{\epsilon}_0\right) \\ &= \sqrt{\mu\varepsilon} \hat{\mathbf{k}} \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^*. \end{aligned}$$

Logo,

$$\langle u \rangle = \frac{\varepsilon}{4} \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^* + \frac{1}{4\mu} \mu\varepsilon \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^* = \frac{\varepsilon}{2} \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^*$$

e

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu} \sqrt{\mu\varepsilon} \hat{\mathbf{k}} \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^* = \frac{1}{2} \sqrt{\frac{\varepsilon}{\mu}}\, \hat{\mathbf{k}} \boldsymbol{\epsilon}_0 \cdot \boldsymbol{\epsilon}_0^*.$$

É interessante notarmos que, a partir dessas equações, também temos

$$\langle \mathbf{S} \rangle = \langle u \rangle \frac{\hat{\mathbf{k}}}{\sqrt{\mu\varepsilon}},$$

onde

$$\frac{\hat{\mathbf{k}}}{\sqrt{\mu\varepsilon}}$$

é a velocidade com que as ondas se propagam. Essa relação é análoga à relação entre a densidade de corrente e a densidade de carga:

$$\mathbf{J} = \rho \mathbf{v}.$$

Quando o meio dielétrico possui uma condutividade g finita, isto é, é um meio condutor, temos

$$\mathbf{J} = g\mathbf{E}.$$

Nesse caso, a equação de onda para $\mathbf{E}$ é dada por:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu g \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0}.$$

Também temos

$$\nabla^2 \mathbf{B} - \mu\varepsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} - \mu g \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0}.$$

As respectivas versões complexas dessas equações de onda são

$$\nabla^2 \boldsymbol{\epsilon} - \mu\varepsilon \frac{\partial^2 \boldsymbol{\epsilon}}{\partial t^2} - \mu g \frac{\partial \boldsymbol{\epsilon}}{\partial t} = \mathbf{0}$$

e

$$\nabla^2 \boldsymbol{\beta} - \mu\varepsilon \frac{\partial^2 \boldsymbol{\beta}}{\partial t^2} - \mu g \frac{\partial \boldsymbol{\beta}}{\partial t} = \mathbf{0}.$$

Se tomamos o ansatz

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_0 \exp\left(i\mathbf{k}\cdot\mathbf{r} - i\omega t\right),$$

obtemos

$$-\mathbf{k}\cdot\mathbf{k} + \mu\varepsilon\omega^2 + i\mu g\omega = 0.$$

Assim, se ω é real, segue que $\mathbf{k}$ é um vetor complexo. Dessa forma, escrevemos

$$\mathbf{k} = \mathbf{k}_r + i\mathbf{k}_i,$$

onde

$$\begin{aligned} \mathbf{k}_r &= \mathrm{Re}\left(\mathbf{k}\right), \\ \mathbf{k}_i &= \mathrm{Im}\left(\mathbf{k}\right). \end{aligned}$$

Portanto,

$$\begin{aligned} \boldsymbol{\epsilon} &= \boldsymbol{\epsilon}_0 \exp\left(-\mathbf{k}_i\cdot\mathbf{r}\right)\exp\left(i\mathbf{k}_r\cdot\mathbf{r} - i\omega t\right), \\ \boldsymbol{\beta} &= \boldsymbol{\beta}_0 \exp\left(-\mathbf{k}_i\cdot\mathbf{r}\right)\exp\left(i\mathbf{k}_r\cdot\mathbf{r} - i\omega t\right). \end{aligned}$$

Portanto,

$$\langle u\rangle = \frac{\varepsilon}{4}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^* + \frac{1}{4\mu}\boldsymbol{\beta}\cdot\boldsymbol{\beta}^*$$

e

$$\langle \mathbf{S}\rangle = \frac{1}{2\mu}\mathrm{Re}\left(\boldsymbol{\epsilon}\times\boldsymbol{\beta}^*\right).$$

Para verificarmos essas relações, façamos:

$$\begin{aligned} \mathrm{Re}\left(\boldsymbol{\epsilon}\right)\cdot\mathrm{Re}\left(\boldsymbol{\epsilon}\right) &= \frac{\boldsymbol{\epsilon}+\boldsymbol{\epsilon}^*}{2}\cdot\frac{\boldsymbol{\epsilon}+\boldsymbol{\epsilon}^*}{2} \\ &= \frac{1}{4}\left(\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^* + \boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}^*\right). \end{aligned}$$

Logo,

$$\langle \mathrm{Re}\left(\boldsymbol{\epsilon}\right)\cdot\mathrm{Re}\left(\boldsymbol{\epsilon}\right)\rangle = \frac{1}{2}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^*.$$

Analogamente,

$$\langle \mathrm{Re}\left(\boldsymbol{\beta}\right)\cdot\mathrm{Re}\left(\boldsymbol{\beta}\right)\rangle = \frac{1}{2}\boldsymbol{\beta}\cdot\boldsymbol{\beta}^*.$$

Com isso, obtemos

$$\langle u\rangle = \frac{\varepsilon}{4}\boldsymbol{\epsilon}\cdot\boldsymbol{\epsilon}^* + \frac{1}{4\mu}\boldsymbol{\beta}\cdot\boldsymbol{\beta}^*.$$

Finalmente, para o fluxo de energia, temos

$$\begin{aligned}\mathrm{Re}\,(\boldsymbol{\epsilon}) \times \mathrm{Re}\,(\boldsymbol{\beta}) &= \frac{\boldsymbol{\epsilon}+\boldsymbol{\epsilon}^*}{2} \times \frac{\boldsymbol{\beta}+\boldsymbol{\beta}^*}{2} \\ &= \frac{1}{4}\left(\boldsymbol{\epsilon}\times\boldsymbol{\beta}+\boldsymbol{\epsilon}\times\boldsymbol{\beta}^*+\boldsymbol{\epsilon}^*\times\boldsymbol{\beta}+\boldsymbol{\epsilon}^*\times\boldsymbol{\beta}^*\right)\end{aligned}$$

e, portanto,

$$\begin{aligned}\langle \mathrm{Re}\,(\boldsymbol{\epsilon}) \times \mathrm{Re}\,(\boldsymbol{\beta})\rangle &= \frac{1}{4}\left(\boldsymbol{\epsilon}\times\boldsymbol{\beta}^*+\boldsymbol{\epsilon}^*\times\boldsymbol{\beta}\right) \\ &= \frac{1}{2}\mathrm{Re}\left(\boldsymbol{\epsilon}\times\boldsymbol{\beta}^*\right).\end{aligned}$$

Logo,

$$\langle \mathbf{S}\rangle = \frac{1}{2\mu}\mathrm{Re}\left(\boldsymbol{\epsilon}\times\boldsymbol{\beta}^*\right).$$