Integral da gaussiana

De Nerdyard

Meu propósito aqui é mostrar que, para $\alpha \in \mathbb{R}, \alpha > 0$

$$\int_{-\infty}^{+\infty} dx \, \exp\left(-\alpha x^2\right) = \sqrt{\frac{\pi}{\alpha}}.$$

Comecemos definindo

$$I = \int_{-\infty}^{+\infty} dx \, \exp\left(-\alpha x^2\right).$$

Então, podemos elevar I ao quadrado e obter

$$I^2 = \left[\int_{-\infty}^{+\infty} dx \, \exp\left(-\alpha x^2\right)\right]^2.$$

Mas,

$$\left[\int_{-\infty}^{+\infty} dx \, \exp\left(-\alpha x^2\right)\right]^2 = \left[\int_{-\infty}^{+\infty} dx \, \exp\left(-\alpha x^2\right)\right]\left[\int_{-\infty}^{+\infty} dy \, \exp\left(-\alpha y^2\right)\right]$$
$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \, dy \, \exp\left(-\alpha x^2\right) \exp\left(-\alpha y^2\right).$$

Assim, podemos escrever que

$$I^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx \, dy \, \exp\left[-\alpha\left(x^2 + y^2\right)\right].$$

Em coordenadas polares, temos

$$r^2 = x^2 + y^2,$$

$$x = r \cos\theta$$

e

$$y = r \,\mathrm{sen}\,\theta.$$

Com isso, o elemento de área no plano xy é dado por

$$dA = dx\,dy \;\; = J\,dr\,d\theta = r\,dr\,d\theta$$

cujo jacobiano é

$$J = \det \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

e, portanto,

$$I^2 = \int_0^{2\pi} d\theta \int_0^{\infty} dr\,r\exp\left(-\alpha r^2\right) = 2\pi \int_0^{\infty} dr\,r\exp\left(-\alpha r^2\right).$$

Note que

$$\int_0^{\infty} dr\,r\exp\left(-\alpha r^2\right) = \frac{1}{2}\int_0^{\infty} d\left(r^2\right)\,\exp\left(-\alpha r^2\right) \; = -\frac{1}{2\alpha}\exp\left(-\alpha r^2\right)\Big|_0^{\infty} = \frac{1}{2\alpha}.$$

Logo,

$$I^2 = 2\pi\frac{1}{2\alpha} = \frac{\pi}{\alpha}.$$

Extraindo a raiz quadrada de ambos os membros dessa equação, obtemos o resultado enunciado acima:

$I = \sqrt{\frac{\pi}{\alpha}}$; *quod erat demonstrandum!*

Obtida de "http://nerdyard.com/wiki/Integral_da_gaussiana"