Uma integral cabeluda

No contexto de minhas pesquisas em sistemas quânticos abertos, há uma integral bem difícil, em princípio, para ser resolvida. Essa integral aparece em vários artigos que consideram ruídos em sistemas de qubits e também em problemas quânticos dissipativos, como no caso da emissão espontânea de fótons em uma cavidade de temperatura finita. Minha intenção aqui não é a descrição do aparecimento dessa integral, mas apenas mostrar sua solução exata. A integral é a seguinte:

$$\Xi = 2\int_0^\infty d\omega \exp\left(-\frac{\omega}{\omega_c}\right)\frac{1-\cos(\omega t)}{\omega}\frac{1}{\exp\left(\frac{\omega}{\omega_T}\right)-1},$$

onde ω_c e ω_T são parâmetros reais positivos. É um fato que essa expressão para Ξ também pode ser escrita assim:

$$\Xi = 2\int_0^t dt' \int_0^\infty d\omega \exp\left(-\frac{\omega}{\omega_c}\right)\sin(\omega t')\frac{1}{\exp\left(\frac{\omega}{\omega_T}\right)-1}.$$

Mas,

$$\begin{aligned}
\frac{1}{\exp\left(\frac{\omega}{\omega_T}\right)-1} &= \frac{\exp\left(-\frac{\omega}{\omega_T}\right)}{1-\exp\left(-\frac{\omega}{\omega_T}\right)} \\
&= \sum_{n=0}^\infty \exp\left(-\frac{\omega}{\omega_T}\right)\exp\left(-n\frac{\omega}{\omega_T}\right) \\
&= \sum_{n=0}^\infty \exp\left[-(n+1)\frac{\omega}{\omega_T}\right],
\end{aligned}$$

isto é,

$$\frac{1}{\exp\left(\frac{\omega}{\omega_T}\right)-1} = \sum_{n=1}^\infty \exp\left(-n\frac{\omega}{\omega_T}\right).$$

Usando essa igualdade, podemos escrever Ξ como

$$\Xi = 2\int_0^t dt' \int_0^\infty d\omega \exp\left(-\frac{\omega}{\omega_c}\right)\sin(\omega t')\sum_{n=1}^\infty \exp\left(-n\frac{\omega}{\omega_T}\right),$$

isto é,

$$\Xi = 2\sum_{n=1}^\infty\int_0^t dt' \int_0^\infty d\omega \exp\left(-\frac{\omega}{\omega_c}\right)\sin(\omega t')\exp\left(-n\frac{\omega}{\omega_T}\right).$$

Como agora não há denominador singular no integrando, podemos também escrever

$$\Xi = 2\sum_{n=1}^{\infty}\int_0^t dt'\mathrm{Im}\left[\int_0^{\infty} d\omega \exp\left(-\frac{\omega}{\omega_c}\right)\exp\left(i\omega t'\right)\exp\left(-n\frac{\omega}{\omega_T}\right)\right],$$

isto é,

$$\Xi = 2\sum_{n=1}^{\infty}\int_0^t dt'\mathrm{Im}\left[\int_0^{\infty} d\omega \exp\left(i\omega t' - n\frac{\omega}{\omega_T} - \frac{\omega}{\omega_c}\right)\right],$$

ou seja,

$$\Xi = 2\sum_{n=1}^{\infty}\int_0^t dt'\mathrm{Im}\left\{\int_0^{\infty} d\omega \exp\left[\left(it' - n\frac{1}{\omega_T} - \frac{1}{\omega_c}\right)\omega\right]\right\}.$$

Como

$$\begin{aligned}\int_0^{\infty} d\omega \exp\left[\left(it' - n\frac{1}{\omega_T} - \frac{1}{\omega_c}\right)\omega\right] &= \frac{-1}{it' - n\frac{1}{\omega_T} - \frac{1}{\omega_c}} \\ &= \frac{\omega_T}{-i\omega_T t' + n + \frac{\omega_T}{\omega_c}},\end{aligned}$$

obtemos

$$\Xi = 2\sum_{n=1}^{\infty}\int_0^t dt'\mathrm{Im}\left(\frac{\omega_T}{-i\omega_T t' + n + \frac{\omega_T}{\omega_c}}\right).$$

O integrando fica

$$\begin{aligned}\mathrm{Im}\left(\frac{\omega_T}{-i\omega_T t' + n + \frac{\omega_T}{\omega_c}}\right) &= \mathrm{Im}\left[\frac{\omega_T\left(i\omega_T t' + n + \frac{\omega_T}{\omega_c}\right)}{\left(-i\omega_T t' + n + \frac{\omega_T}{\omega_c}\right)\left(i\omega_T t' + n + \frac{\omega_T}{\omega_c}\right)}\right] \\ &= \mathrm{Im}\left[\frac{\omega_T\left(i\omega_T t' + n + \frac{\omega_T}{\omega_c}\right)}{\left(n + \frac{\omega_T}{\omega_c}\right)^2 + \left(\omega_T t'\right)^2}\right],\end{aligned}$$

isto é,

$$\mathrm{Im}\left(\frac{\omega_T}{-i\omega_T t' + n + \frac{\omega_T}{\omega_c}}\right) = \frac{\omega_T^2 t'}{\left(n + \frac{\omega_T}{\omega_c}\right)^2 + \left(\omega_T t'\right)^2}.$$

Então,

$$\begin{aligned}\Xi &= 2\sum_{n=1}^{\infty}\int_0^t dt'\frac{\omega_T^2 t'}{\left(n + \frac{\omega_T}{\omega_c}\right)^2 + \left(\omega_T t'\right)^2} \\ &= \sum_{n=1}^{\infty}\left\{\ln\left[\left(n + \frac{\omega_T}{\omega_c}\right)^2 + (\omega_T t)^2\right] - \ln\left[\left(n + \frac{\omega_T}{\omega_c}\right)^2\right]\right\},\end{aligned}$$

isto é,

$$\Xi = \sum_{n=1}^{\infty}\left\{\ln\left[\left|n+\frac{\omega_T}{\omega_c}+i\omega_T t\right|^2\right]-\ln\left(n^2\right)-\ln\left[\left(n+\frac{\omega_T}{\omega_c}\right)^2\right]+\ln\left(n^2\right)\right\},$$

ou seja,

$$\Xi = \sum_{n=1}^{\infty}\left\{\ln\left[\left|1+\frac{\frac{\omega_T}{\omega_c}+i\omega_T t}{n}\right|^2\right]-2\ln\left(1+\frac{\frac{\omega_T}{\omega_c}}{n}\right)\right\}.$$

Para simplificar a notação, seja

$$z = \frac{\omega_T}{\omega_c}+i\omega_T t$$

e

$$a = \frac{\omega_T}{\omega_c}.$$

Então, note que

$$\begin{aligned}\Xi &= \sum_{n=1}^{\infty}\ln\left(\left|1+\frac{z}{n}\right|^2\right)-2\sum_{n=1}^{\infty}\ln\left(1+\frac{a}{n}\right)\\ &= \ln\left(\prod_{n=1}^{\infty}\left|1+\frac{z}{n}\right|^2\right)-2\ln\left[\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right)\right],\end{aligned}$$

isto é,

$$\begin{aligned}\Xi &= \ln\left\{\left[\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right)\right]\left[\prod_{n=1}^{\infty}\left(1+\frac{z^*}{n}\right)\exp\left(-\frac{z^*}{n}\right)\right]\left[\prod_{k=1}^{\infty}\exp\left(2\frac{a}{k}\right)\right]\right\}\\ &- \ln\left\{\left[\prod_{m=1}^{\infty}\left(1+\frac{a}{m}\right)\exp\left(-\frac{a}{m}\right)\right]\left[\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right)\exp\left(-\frac{a}{n}\right)\right]\left[\prod_{k=1}^{\infty}\exp\left(2\frac{a}{k}\right)\right]\right\},\end{aligned}$$

ou ainda,

$$\begin{aligned}\Xi &= \ln\left\{\left[\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right)\right]\left[\prod_{n=1}^{\infty}\left(1+\frac{z^*}{n}\right)\exp\left(-\frac{z^*}{n}\right)\right]\right\}\\ &- \ln\left\{\left[\prod_{m=1}^{\infty}\left(1+\frac{a}{m}\right)\exp\left(-\frac{a}{m}\right)\right]\left[\prod_{n=1}^{\infty}\left(1+\frac{a}{n}\right)\exp\left(-\frac{a}{n}\right)\right]\right\}.\end{aligned}$$

Pela definição de Weierstrass da função Γ, temos

$$\frac{1}{\Gamma(z)} = z\exp(\gamma z)\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right),$$

isto é,

$$\frac{1}{z\Gamma\left(z\right)} = \exp\left(\gamma z\right)\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right),$$

ou seja,

$$\frac{1}{\Gamma\left(z+1\right)} = \exp\left(\gamma z\right)\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right),$$

ou ainda,

$$\frac{1}{z!} = \exp\left(\gamma z\right)\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right).$$

Então,

$$\prod_{m=1}^{\infty}\left(1+\frac{z}{m}\right)\exp\left(-\frac{z}{m}\right) = \frac{\exp\left(-\gamma z\right)}{z!}.$$

Portanto,

$$\begin{aligned}\Xi &= \ln\left[\frac{\exp\left(-\gamma z\right)}{z!}\frac{\exp\left(-\gamma z^*\right)}{z^*!}\right]\\ &- \ln\left[\frac{\exp\left(-\gamma a\right)}{a!}\frac{\exp\left(-\gamma a\right)}{a!}\right].\end{aligned}$$

Logo, lembrando que

$$z = a + i\omega_T t$$

e

$$a = \frac{\omega_T}{\omega_c},$$

vem

$$\Xi = \ln\left[\frac{1}{\left|z!\right|^2}\right] - \ln\left[\frac{1}{\left(a!\right)^2}\right],$$

isto é,

$$\Xi = \ln\left[\frac{\left(a!\right)^2}{\left|z!\right|^2}\right],$$

ou seja,

$$\Xi = 2\ln\left(\frac{a!}{\left|z!\right|}\right),$$

ou ainda,

$$\Xi \quad = \quad 2\ln\left[\frac{\left(\frac{\omega_T}{\omega_c}\right)!}{\left|\left(\frac{\omega_T}{\omega_c}+i\omega_T t\right)!\right|}\right].$$

Assim, a integral cabeluda é exatamente dada por

$$2\int_0^\infty d\omega \exp\left(-\frac{\omega}{\omega_c}\right)\frac{1-\cos\left(\omega t\right)}{\omega}\frac{1}{\exp\left(\frac{\omega}{\omega_T}\right)-1} \quad = \quad 2\ln\left[\frac{\left(\frac{\omega_T}{\omega_c}\right)!}{\left|\left(\frac{\omega_T}{\omega_c}+i\omega_T t\right)!\right|}\right].$$