Emissão de momentum angular por uma partícula carregada em movimento arbitrário

Há um problema no livor de J. D. Jackson, que é o problema número 6.11 da segunda edição ou o 6.10, da terceira, onde o momentum angular do campo eletromagnético é examinado. Em resumo, o balanço do momentum angular pode ser expresso pela equação:

$$\frac{d}{dt}\left(\mathbf{L}_{\text{mecânico}}+\mathbf{L}_{\text{campos}}\right) \quad = \quad -\oint_{S(V)} da\,\hat{\mathbf{n}}\cdot\overleftrightarrow{M},$$

onde $\mathbf{L}_{\text{mecânico}}$ é o momentum angular da matéria carregada,

$$\mathbf{L}_{\text{campos}} \quad = \quad \frac{1}{4\pi c}\int_V d^3r\,\mathbf{r}\times(\mathbf{E}\times\mathbf{B})$$

é o momentum angular dos campos armazenado na região V,

$$\overleftrightarrow{M} \quad = \quad \overleftrightarrow{T}\times\mathbf{r}$$

é o fluxo do momentum angular atravessando a superfície $S(V)$, fronteira de V e

$$\overleftrightarrow{T} \quad = \quad \frac{1}{4\pi}\left[\mathbf{EE}+\mathbf{BB}-\frac{\overleftrightarrow{I}}{2}\left(\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B}\right)\right]$$

é o fluxo de momentum linear através de $S(V)$.

Os campos $\mathbf{E}$ e $\mathbf{B}$ produzidos por uma carga q que descreve uma trajetória $\mathbf{r}_0(t)$ arbitrária são deduzidos a partir dos potenciais de Liénard-Wiechert:

$$\mathbf{E}(\mathbf{r},t) \quad = \quad q\left[\frac{\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\left(1-\beta^2\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}+\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right]$$

e

$$\mathbf{B}(\mathbf{r},t) \quad = \quad \hat{\mathbf{R}}\times\mathbf{E}(\mathbf{r},t),$$

onde

$$\mathbf{R} \quad = \quad \mathbf{r}-\mathbf{r}_0\left(t_{\text{ret}}\right),$$

$$t_{\text{ret}} \quad = \quad t-\frac{\left|\mathbf{r}-\mathbf{r}_0\left(t_{\text{ret}}\right)\right|}{c},$$

$$R \quad = \quad |\mathbf{R}|,$$

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R},$$

$$\begin{aligned} \mathbf{v} &= \mathbf{v}\left(t_{\text{ret}}\right) \\ &= \frac{d\mathbf{r}_0\left(t_{\text{ret}}\right)}{dt_{\text{ret}}} \\ &= \dot{\mathbf{r}}_0\left(t_{\text{ret}}\right) \end{aligned}$$

e

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}.$$

Seja $d^2\mathbf{L}/d\Omega$ o momentum angular por unidade de ângulo sólido $d\Omega$ emitido pela partícula durante o intervalo de tempo dt_{ret}. Seja S a superfície esférica, centrada em $\mathbf{r}_0\left(t_{\text{ret}}\right)$, com raio R. Dentro do ângulo sólido $d\Omega$, em torno do vetor $\mathbf{R}$, o momentum angular $d^2\mathbf{L}$ leva um intervalo de tempo dt para passar através do elemento de área $R^2 d\Omega$. A equação

$$\frac{d}{dt}\left(\mathbf{L}_{\text{mecânico}} + \mathbf{L}_{\text{campos}}\right) = -\oint_{S(V)} da\,\hat{\mathbf{n}} \cdot \overleftrightarrow{M},$$

que expressa o balanço do momentum angular, fornece o momentum angular emitido, através de S, por unidade de tempo e por elemento de ângulo sólido:

$$\frac{d^2\mathbf{L}}{dt d\Omega} = \frac{R^3}{4\pi}\left[\left(\hat{\mathbf{R}} \cdot \mathbf{E}\right)\left(\mathbf{E} \times \hat{\mathbf{R}}\right) + \left(\hat{\mathbf{R}} \cdot \mathbf{B}\right)\left(\mathbf{B} \times \hat{\mathbf{R}}\right)\right].$$

O sinal positivo expressa o fato de que $d^2\mathbf{L}$ é o momentum angular que a carga emite. Como

$$\mathbf{B} = \hat{\mathbf{R}} \times \mathbf{E},$$

segue que

$$\hat{\mathbf{R}} \cdot \mathbf{B} = 0$$

e, portanto,

$$\frac{d^2\mathbf{L}}{dt d\Omega} = \frac{R^3}{4\pi}\left(\hat{\mathbf{R}} \cdot \mathbf{E}\right)\left(\mathbf{E} \times \hat{\mathbf{R}}\right).$$

Procedendo de forma análoga ao caso da emissão de energia e da emissão de momentum linear, escrevemos

$$\frac{d\mathbf{L}}{dt_{\text{ret}}} = \frac{1}{4\pi}\int_{4\pi} d\Omega \frac{dt}{dt_{\text{ret}}} R^3 \left(\hat{\mathbf{R}} \cdot \mathbf{E}\right)\left(\mathbf{E} \times \hat{\mathbf{R}}\right).$$

Notemos que no integrando aparece R^3, ao invés de R^2, e, portanto, não podemos utilizar apenas o campo elétrico de radiação, mas o campo elétrico com todos os termos, isto é,

$$\mathbf{E}\left(\mathbf{r}, t\right) = q\left[\frac{\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right)\left(1 - \beta^2\right)}{R^2\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)^3} + \frac{\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}} - \boldsymbol{\beta}\right) \times \mathbf{a}\right]}{Rc^2\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)^3}\right].$$

Logo,

$$
\begin{aligned}
\hat{\mathbf{R}}\cdot\mathbf{E} &= q\frac{\hat{\mathbf{R}}\cdot\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\left(1-\beta^2\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\\
&= q\frac{\left(1-\beta^2\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}
\end{aligned}
$$

e

$$
\begin{aligned}
\mathbf{E}\times\hat{\mathbf{R}} &= q\frac{\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\hat{\mathbf{R}}\left(1-\beta^2\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}+q\frac{\left\{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right\}\times\hat{\mathbf{R}}}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\\
&= q\frac{\left(\hat{\mathbf{R}}\times\boldsymbol{\beta}\right)\left(1-\beta^2\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}+q\frac{\left(\hat{\mathbf{R}}\times\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)+\left(\hat{\mathbf{R}}\times\mathbf{a}\right)\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}.
\end{aligned}
$$

Com esses resultados, até ordem R^{-3}, obtemos

$$
\left(\hat{\mathbf{R}}\cdot\mathbf{E}\right)\left(\mathbf{E}\times\hat{\mathbf{R}}\right) \approx \frac{q^2}{c^2}\left[\frac{\left(1-\beta^2\right)\left(\hat{\mathbf{R}}\times\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)}{R^3\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5}+\frac{\left(1-\beta^2\right)\left(\hat{\mathbf{R}}\times\mathbf{a}\right)}{R^3\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4}\right].
$$

Da expressão para o tempo retardado,

$$
t_{\text{ret}} = t-\frac{\left|\mathbf{r}-\mathbf{r}_0\left(t_{\text{ret}}\right)\right|}{c},
$$

segue que

$$
\frac{dt}{dt_{\text{ret}}} = 1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}
$$

e, portanto,

$$
\frac{d\mathbf{L}}{dt_{\text{ret}}} = \frac{q^2\left(1-\beta^2\right)}{4\pi c^2}\int_{4\pi}d\Omega\left[\frac{\left(\hat{\mathbf{R}}\times\boldsymbol{\beta}\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4}+\frac{\left(\hat{\mathbf{R}}\times\mathbf{a}\right)}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right].
$$

Como no caso da emissão de energia, encontramos

$$
\begin{aligned}
\int_{4\pi}d\Omega\frac{\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} &= \frac{1}{2}\frac{\partial}{\partial\boldsymbol{\beta}}\int_{4\pi}d\Omega\frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\\
&= \frac{1}{2}\frac{\partial}{\partial\boldsymbol{\beta}}\frac{4\pi}{\left(1-\beta^2\right)}\\
&= \frac{4\pi\boldsymbol{\beta}}{\left(1-\beta^2\right)^2}
\end{aligned}
$$

e

$$\begin{aligned}
\int_{4\pi} d\Omega \frac{\hat{\mathbf{R}}\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} &= \frac{1}{6}\frac{\partial}{\partial\boldsymbol{\beta}}\frac{\partial}{\partial\boldsymbol{\beta}}\int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} \\
&= \frac{1}{6}\frac{\partial}{\partial\boldsymbol{\beta}}\frac{\partial}{\partial\boldsymbol{\beta}}\frac{4\pi}{(1-\beta^2)} \\
&= \frac{1}{6}\frac{\partial}{\partial\boldsymbol{\beta}}\left[\frac{8\pi\boldsymbol{\beta}}{(1-\beta^2)^2}\right],
\end{aligned}$$

isto é,

$$\int_{4\pi} d\Omega \frac{\hat{\mathbf{R}}\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \frac{4\pi \overleftrightarrow{I}}{3(1-\beta^2)^2}+\frac{16\pi\boldsymbol{\beta}\boldsymbol{\beta}}{3(1-\beta^2)^3},$$

Com essas integrais calculadas, a taxa de emissão de momentum angular fica

$$\begin{aligned}
\frac{d\mathbf{L}}{dt_{\text{ret}}} &= \frac{q^2\left(1-\beta^2\right)}{4\pi c^2}\left[\frac{4\pi\mathbf{a}\cdot\overleftrightarrow{I}\times\boldsymbol{\beta}}{3(1-\beta^2)^2}+\frac{16\pi\mathbf{a}\cdot\boldsymbol{\beta}\boldsymbol{\beta}\times\boldsymbol{\beta}}{3(1-\beta^2)^3}+\frac{4\pi\boldsymbol{\beta}\times\mathbf{a}}{(1-\beta^2)^2}\right] \\
&= \frac{q^2}{c^2}\left[\frac{\mathbf{a}\times\boldsymbol{\beta}}{3(1-\beta^2)}+\frac{\boldsymbol{\beta}\times\mathbf{a}}{(1-\beta^2)}\right]
\end{aligned}$$

e, portanto,

$$\frac{d\mathbf{L}}{dt_{\text{ret}}} = \frac{2q^2}{3c^2}\left[\frac{\boldsymbol{\beta}\times\mathbf{a}}{(1-\beta^2)}\right].$$