Os potenciais de Liénard-Wiechert

Vou introduzir os cálculos dos potenciais de Liénard-Wiechert, escalar e vetorial, de uma partícula carregada executando um movimento com trajetória dada. Como queremos os campos causados pela partícula, utilizamos as soluções retardadas dos potenciais no calibre de Lorentz:

$$\phi(\mathbf{r},t) = \int_{V_\infty} d^3r' \, \frac{\rho\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}$$

e

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int_{V_\infty} d^3r' \, \frac{\mathbf{J}\left(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

Para uma carga q descrevendo uma trajetória $\mathbf{r}_0(t)$, temos

$$\rho(\mathbf{r}',t') = q\delta^{(3)}(\mathbf{r}'-\mathbf{r}_0(t'))$$

e

$$\mathbf{J}(\mathbf{r}',t') = q\mathbf{v}(t')\,\delta^{(3)}(\mathbf{r}'-\mathbf{r}_0(t')),$$

para quaisquer $\mathbf{r}'$ e t', onde

$$\mathbf{v}(t') = \frac{d\mathbf{r}_0(t')}{dt'}.$$

Assim, o potencial escalar, por exemplo, fica

$$\phi(\mathbf{r},t) = \int_{V_\infty} d^3r' \, \frac{q\delta^{(3)}\left(\mathbf{r}'-\mathbf{r}_0\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

O problema é integrarmos

$$\int_{V_\infty} d^3r' \, \frac{\delta^{(3)}\left(\mathbf{r}'-\mathbf{r}_0\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{|\mathbf{r}-\mathbf{r}'|}.$$

Para tal proeza, há um truque: é óbvio que a integral acima pode ser escrita como

$$\int_{-\infty}^{+\infty} dt' \int_{V_\infty} d^3r' \, \frac{\delta^{(3)}(\mathbf{r}'-\mathbf{r}_0(t'))}{|\mathbf{r}-\mathbf{r}'|}\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)$$

e, portanto, integrando sobre a variável $\mathbf{r}'$, obtemos

$$\int_{V_\infty} d^3r' \, \frac{\delta^{(3)}\left(\mathbf{r}'-\mathbf{r}_0\left(t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}\right)\right)}{|\mathbf{r}-\mathbf{r}'|} = \int_{-\infty}^{+\infty} dt' \frac{\delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}\right)}{|\mathbf{r}-\mathbf{r}_0(t')|}.$$

A seguir, tomamos como fixos $\mathbf{r}$ e t e introduzimos a função

$$f(t') = t'-t+\frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}.$$

Portanto, temos a seguinte integral para calcular:

$$\int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r}-\mathbf{r}_0(t')|}.$$

Das propriedades da função delta de Dirac, podemos escrever

$$\int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r}-\mathbf{r}_0(t')|} = \sum_k \int_{-\infty}^{+\infty} \frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|} \frac{\delta(t'-t_k)}{\left|\frac{df(t')}{dt'}\right|_{t'=t_k}},$$

onde os t_k's são os instantes de tempo em que $f(t_k)=0$. Em outras palavras, queremos os instantes de tempo t_k's tais que

$$t_k = t - \frac{|\mathbf{r}-\mathbf{r}_0(t_k)|}{c}.$$

O fato é que só há um instante de tempo para essa relação valer. Para vermos isso, suponhamos que existam dois instantes, t_1 e t_2, com $t_1 \neq t_2$, tais que

$$t_1 = t - \frac{|\mathbf{r}-\mathbf{r}_0(t_1)|}{c}$$

e

$$t_2 = t - \frac{|\mathbf{r}-\mathbf{r}_0(t_2)|}{c}.$$

Sem perda de generalidade, suponhamos que $t_2 > t_1$. Logo,

$$c(t_2-t_1) = |\mathbf{r}-\mathbf{r}_0(t_1)| - |\mathbf{r}-\mathbf{r}_0(t_2)|.$$

Mas,

$$\begin{aligned} |\mathbf{r}-\mathbf{r}_0(t_1)| - |\mathbf{r}-\mathbf{r}_0(t_2)| &\leqslant |\mathbf{r}-\mathbf{r}_0(t_1) - (\mathbf{r}-\mathbf{r}_0(t_2))| \\ &= |\mathbf{r}_0(t_2)-\mathbf{r}_0(t_1)|. \end{aligned}$$

Assim,

$$c \leqslant \frac{|\mathbf{r}_0(t_2)-\mathbf{r}_0(t_1)|}{t_2-t_1},$$

implicando que a partícula deveria ir de $\mathbf{r}_0(t_1)$ até $\mathbf{r}_0(t_2)$ com uma velocidade média maior ou igual à velocidade da luz no vácuo. Como, por hipótese, estamos considerando uma partícula massiva, de massa $m>0$, concluímos que há, no máximo, um instante de tempo em que a equação

$$t_k = t - \frac{|\mathbf{r}-\mathbf{r}_0(t_k)|}{c}$$

vale e definimos esse instante como o tempo retardado:

$$t_R = t - \frac{|\mathbf{r}-\mathbf{r}_0(t_R)|}{c}.$$

O resultado da integral acima pode ser escrito em termos do tempo retardado como

$$\begin{aligned} \int_{-\infty}^{+\infty} dt' \frac{\delta(f(t'))}{|\mathbf{r}-\mathbf{r}_0(t')|} &= \int_{-\infty}^{+\infty} \frac{dt'}{|\mathbf{r}-\mathbf{r}_0(t')|} \frac{\delta(t'-t_R)}{\left|\frac{df(t')}{dt'}\right|_{t'=t_R}} \\ &= \frac{1}{|\mathbf{r}-\mathbf{r}_0(t_R)| \left|\frac{df(t')}{dt'}\right|_{t'=t_R}}. \end{aligned}$$

Calculemos, explicitamente, a derivada temporal da função f :

$$\begin{aligned} \frac{df(t')}{dt'} &= \frac{d}{dt'}\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}_0(t')|}{c}\right) \\ &= 1+\frac{1}{c}\frac{d\,|\mathbf{r}-\mathbf{r}_0(t')|}{dt'}. \end{aligned}$$

Notemos que

$$\begin{aligned}
\frac{d\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|}{dt'} &= \frac{1}{2\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|}\frac{d\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|^2}{dt'} \\
&= \frac{1}{2\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|}\frac{d\left[\mathbf{r}-\mathbf{r}_0\left(t'\right)\right]\cdot\left[\mathbf{r}-\mathbf{r}_0\left(t'\right)\right]}{dt'} \\
&= \frac{\left[\mathbf{r}-\mathbf{r}_0\left(t'\right)\right]}{\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|}\cdot\frac{d\left[\mathbf{r}-\mathbf{r}_0\left(t'\right)\right]}{dt'} \\
&= -\frac{\left[\mathbf{r}-\mathbf{r}_0\left(t'\right)\right]}{\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|}\cdot\frac{d\mathbf{r}_0\left(t'\right)}{dt'} \\
&= -\frac{\left[\mathbf{r}-\mathbf{r}_0\left(t'\right)\right]}{\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|}\cdot\mathbf{v}\left(t'\right).
\end{aligned}$$

Portanto,

$$\int_{-\infty}^{+\infty} dt' \frac{\delta\left(f\left(t'\right)\right)}{\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|} = \frac{1}{\left|\left|\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right|-\left[\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right]\cdot\frac{\mathbf{v}\left(t_R\right)}{c}\right|}.$$

Como $|\mathbf{v}/c|<1$, temos

$$\int_{-\infty}^{+\infty} dt' \frac{\delta\left(f\left(t'\right)\right)}{\left|\mathbf{r}-\mathbf{r}_0\left(t'\right)\right|} = \frac{1}{\left|\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right|-\left[\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right]\cdot\frac{\mathbf{v}\left(t_R\right)}{c}}$$

e os potenciais podem ser escritos como

$$\phi\left(\mathbf{r},t\right) = \frac{q}{\left|\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right|-\left[\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right]\cdot\frac{\mathbf{v}\left(t_R\right)}{c}}$$

e, analogamente,

$$\mathbf{A}\left(\mathbf{r},t\right) = \frac{q\mathbf{v}\left(t_R\right)}{c\left|\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right|-\left[\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right]\cdot\mathbf{v}\left(t_R\right)},$$

onde

$$t_R = t-\frac{\left|\mathbf{r}-\mathbf{r}_0\left(t_R\right)\right|}{c}.$$

Esses são os chamados potenciais de Liénard-Wiechert. Para simplificar a notação, definamos

$$\begin{aligned}
\mathbf{R} &= \mathbf{r}-\mathbf{r}_0\left(t_R\right), \\
R &= \left|\mathbf{R}\right|, \\
\hat{\mathbf{R}} &= \frac{\mathbf{R}}{R}, \\
\mathbf{v} &= \mathbf{v}\left(t_R\right) \\
&= \frac{d\mathbf{r}_0\left(t_R\right)}{dt_R} \\
&= \dot{\mathbf{r}}_0\left(t_R\right)
\end{aligned}$$

e

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}.$$

Com essas definições, podemos simplificar as expressões dos potenciais assim:

$$\begin{aligned}
\phi\left(\mathbf{r},t\right) &= \frac{q}{R-\mathbf{R}\cdot\boldsymbol{\beta}}, \\
\mathbf{A}\left(\mathbf{r},t\right) &= \frac{q\boldsymbol{\beta}}{R-\mathbf{R}\cdot\boldsymbol{\beta}}
\end{aligned}$$

e

$$t_R = t-\frac{R}{c}.$$

Calculando o Campo Indução Magnética e o Campo Elétrico

Vamos agora calcular os campos $\mathbf{B}$ e $\mathbf{E}$. Comecemos com o cálculo de $\mathbf{B}$:

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$$

ou, em termos de componentes,

$$B_i = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{ijk}\partial_j A_k,$$

onde ε_{ijk} é o tensor de Levi-Civita, que é dado por

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{se } (i,j,k) \text{ for uma permutação par de } (1,2,3)\,, \\ 0, & \text{se pelo menos dois dos índices } i,j,k \text{ forem iguais e} \\ -1, & \text{se } (i,j,k) \text{ for uma permutação ímpar de } (1,2,3)\,. \end{cases}$$

Também utilizemos a notação

$$\partial_j = \frac{\partial}{\partial x_j},$$

para $j = 1, 2, 3$. A convenção de Einstein para somas permite que escrevamos

$$B_i = \varepsilon_{ijk}\partial_j A_k,$$

onde subentendemos que os índices j e k estão somados de 1 a 3, porque aparecem repetidos no mesmo termo. Temos, assim,

$$\begin{aligned} \partial_j A_k &= \frac{q}{c}\partial_j\left(\frac{v_k}{R - \mathbf{R}\cdot\boldsymbol{\beta}}\right) \\ &= \frac{q}{c}\left[\frac{\partial_j v_k}{R - \mathbf{R}\cdot\boldsymbol{\beta}} + v_k\partial_j\left(\frac{1}{R - \mathbf{R}\cdot\boldsymbol{\beta}}\right)\right] \\ &= \frac{q}{c}\left[\frac{\partial_j v_k}{R - \mathbf{R}\cdot\boldsymbol{\beta}} - \frac{v_k}{\left(R - \mathbf{R}\cdot\boldsymbol{\beta}\right)^2}\left(\partial_j R - \boldsymbol{\beta}\cdot\partial_j\mathbf{R} - \mathbf{R}\cdot\partial_j\boldsymbol{\beta}\right)\right]. \end{aligned}$$

Também,

$$\begin{aligned} \partial_j v_k &= \frac{dv_k}{dt_R}\partial_j t_R \\ &= a_k\partial_j t_R, \end{aligned}$$

Façamos agora o cálculo de $\partial_j t_R$:

$$\begin{aligned} \partial_j t_R &= \partial_j\left(t - \frac{R}{c}\right) \\ &= -\frac{1}{c}\partial_j R \\ &= -\frac{1}{c}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j + \hat{\mathbf{R}}\cdot\boldsymbol{\beta}\partial_j t_R, \end{aligned}$$

ou seja,

$$\left(1 - \hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\partial_j t_R = -\frac{1}{c}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j,$$

resultando em

$$\partial_j t_R = -\frac{1}{c}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1 - \hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}.$$

Portanto,

$$\partial_j v_k = -\frac{a_k}{c}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)},$$

$$\begin{aligned}
\partial_j \mathbf{R} &= \partial_j \mathbf{r} - \partial_j \mathbf{r}_0\left(t_R\right) \\
&= \hat{\mathbf{x}}_j - \frac{d\mathbf{r}_0\left(t_R\right)}{dt_R}\partial_j t_R \\
&= \hat{\mathbf{x}}_j + \mathbf{v}\frac{1}{c}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} \\
&= \hat{\mathbf{x}}_j + \boldsymbol{\beta}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)},
\end{aligned}$$

$$\begin{aligned}
\partial_j R &= \frac{1}{2R}\partial_j R^2 \\
&= \frac{1}{2R}\partial_j\left(\mathbf{R}\cdot\mathbf{R}\right) \\
&= \frac{\mathbf{R}}{R}\cdot\partial_j\mathbf{R} \\
&= \hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j + \hat{\mathbf{R}}\cdot\boldsymbol{\beta}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} \\
&= \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)+\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} \\
&= \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j-\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j\hat{\mathbf{R}}\cdot\boldsymbol{\beta}+\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} \\
&= \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}
\end{aligned}$$

e

$$\begin{aligned}
\partial_j\boldsymbol{\beta} &= \frac{1}{c}\partial_j\mathbf{v} \\
&= \frac{1}{c}\frac{d\mathbf{v}}{dt_R}\partial_j t_R \\
&= -\frac{\mathbf{a}}{c^2}\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)},
\end{aligned}$$

onde denotamos

$$a_k = \frac{dv_k}{dt_R}$$

e

$$\mathbf{a} = \frac{d\mathbf{v}}{dt_R}.$$

Assim,

$$\begin{aligned}
\partial_j A_k &= \frac{q}{c}\left[\frac{\partial_j v_k}{R-\mathbf{R}\cdot\boldsymbol{\beta}} - \frac{v_k}{\left(R-\mathbf{R}\cdot\boldsymbol{\beta}\right)^2}\left(\partial_j R - \boldsymbol{\beta}\cdot\partial_j\mathbf{R} - \mathbf{R}\cdot\partial_j\boldsymbol{\beta}\right)\right] \\
&= \frac{q}{c}\left[\frac{\partial_j v_k}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} - \frac{v_k\left(\partial_j R - \boldsymbol{\beta}\cdot\partial_j\mathbf{R} - R\hat{\mathbf{R}}\cdot\partial_j\boldsymbol{\beta}\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\right] \\
&= \frac{q}{c}\left[\frac{\partial_j v_k}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} + \frac{v_k\hat{\mathbf{R}}\cdot\partial_j\boldsymbol{\beta}}{R\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} - \frac{v_k\left(\partial_j R - \boldsymbol{\beta}\cdot\partial_j\mathbf{R}\right)}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\right].
\end{aligned}$$

Substituindo as derivadas parciais, temos

$$\begin{aligned}
\partial_j A_k &= \frac{q}{c}\left[\frac{-a_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{Rc\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} - \frac{v_k\hat{\mathbf{R}}\cdot\mathbf{a}\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right. \\
&- \left.\frac{v_k}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}\left(\frac{\left(1-\beta^2\right)\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_j}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)} - \beta_j\right)\right]
\end{aligned}$$

onde

$$\beta_j = \frac{v_j}{c}.$$

Em termos vetoriais, podemos escrever

$$\begin{aligned}
\mathbf{B} &= \frac{q}{c}\left[\frac{-\hat{\mathbf{R}}\times\mathbf{a}}{Rc\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} - \frac{\hat{\mathbf{R}}\cdot\mathbf{a}\hat{\mathbf{R}}\times\mathbf{v}}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right. \\
&- \left.\frac{\left(1-\beta^2\right)\hat{\mathbf{R}}\times\mathbf{v}}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right],
\end{aligned}$$

onde o termo proporcional a $\boldsymbol{\beta}\times\mathbf{v}$ se anula. Podemos ainda escrever

$$\begin{aligned}
\mathbf{B} &= \frac{q}{c}\hat{\mathbf{R}}\times\left[\frac{-\mathbf{a}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) - \hat{\mathbf{R}}\cdot\mathbf{a}\boldsymbol{\beta}}{Rc\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} - \frac{\left(1-\beta^2\right)\mathbf{v}}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right] \\
&= \frac{q}{c}\hat{\mathbf{R}}\times\left[\frac{-\mathbf{a}\hat{\mathbf{R}}\cdot\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right) + \hat{\mathbf{R}}\cdot\mathbf{a}\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)}{Rc\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} - \frac{\left(1-\beta^2\right)\mathbf{v}}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right],
\end{aligned}$$

onde adicionamos um termo proporcional a $\hat{\mathbf{R}}$ entre colchetes, que não contribui para $\mathbf{B}$, pois é multiplicado vetorialmente pelo $\hat{\mathbf{R}}$ que aparece fora dos colchetes. Notamos agora que

$$-\mathbf{a}\hat{\mathbf{R}}\cdot\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right) + \hat{\mathbf{R}}\cdot\mathbf{a}\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right) = \hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right].$$

Logo,

$$\mathbf{B} = \frac{q}{c}\hat{\mathbf{R}}\times\left\{\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} - \frac{\left(1-\beta^2\right)\mathbf{v}}{R^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right\}.$$

De maneira análoga, calculamos o campo elétrico e obtemos

$$
\begin{aligned}
\mathbf{E} &= q\left[\frac{\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\left(1-\beta^{2}\right)}{R^{2}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^{3}}+\frac{\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\hat{\mathbf{R}}\cdot\mathbf{a}}{Rc^{2}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^{3}}-\frac{\mathbf{a}}{Rc^{2}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^{2}}\right] \\
&= q\left[\frac{\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\left(1-\beta^{2}\right)}{R^{2}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^{3}}+\frac{\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\hat{\mathbf{R}}\cdot\mathbf{a}-\mathbf{a}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)}{Rc^{2}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^{3}}\right] \\
&= q\left[\frac{\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\left(1-\beta^{2}\right)}{R^{2}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^{3}}+\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc^{2}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^{3}}\right]
\end{aligned}
$$

e, portanto,

$$\mathbf{B} = \hat{\mathbf{R}}\times\mathbf{E}.$$