Emissão de momentum por uma partícula carregada em movimento arbitrário

Uma partícula carregada acelerada não irradia apenas energia, pois também emite momentum linear e momentum angular. Aqui vamos analisar a emissão de momentum linear por uma partícula de carga q executando um movimento arbitrário. Os campos de radiação, $\mathbf{E}_{\text{rad}}$ e $\mathbf{B}_{\text{rad}}$, produzidos por uma carga q que descreve uma trajetória $\mathbf{r}_0(t)$ arbitrária são deduzidos a partir dos potenciais de Liénard-Wiechert:

$$\mathbf{E}_{\text{rad}}(\mathbf{r},t) = q\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}$$

e

$$\mathbf{B}_{\text{rad}}(\mathbf{r},t) = \hat{\mathbf{R}}\times\mathbf{E}_{\text{rad}}(\mathbf{r},t),$$

onde

$$\mathbf{R} = \mathbf{r}-\mathbf{r}_0(t_{\text{ret}}),$$

$$t_{\text{ret}} = t-\frac{|\mathbf{r}-\mathbf{r}_0(t_{\text{ret}})|}{c},$$

$$R = |\mathbf{R}|,$$

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R},$$

$$\begin{aligned}\mathbf{v} &= \mathbf{v}(t_{\text{ret}})\\ &= \frac{d\mathbf{r}_0(t_{\text{ret}})}{dt_{\text{ret}}}\\ &= \dot{\mathbf{r}}_0(t_{\text{ret}})\end{aligned}$$

e

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}.$$

Seja $d^2\mathbf{P}/d\Omega$ o momentum por unidade de ângulo sólido $d\Omega$ emitido pela partícula durante o intervalo de tempo dt_{ret}. Seja S a superfície esférica, centrada em $\mathbf{r}_0(t_{\text{ret}})$, com raio R. Dentro do ângulo sólido $d\Omega$, em torno do vetor $\mathbf{R}$, o momentum $d^2\mathbf{P}$ leva um intervalo de tempo dt para passar através do elemento de área $R^2 d\Omega$. A lei de conservação de momentum linear envolvendo campos e matéria fornece a expressão para o momentum emitido, através de S, por unidade de tempo e por elemento de ângulo sólido:

$$\frac{d^2\mathbf{P}}{dtd\Omega} = -\frac{R^2}{4\pi}\left[\left(\hat{\mathbf{R}}\cdot\mathbf{E}_{\mathrm{rad}}\right)\mathbf{E}_{\mathrm{rad}}+\left(\hat{\mathbf{R}}\cdot\mathbf{B}_{\mathrm{rad}}\right)\mathbf{B}_{\mathrm{rad}}-\frac{1}{2}\hat{\mathbf{R}}\left(\mathbf{E}_{\mathrm{rad}}\cdot\mathbf{E}_{\mathrm{rad}}\right)-\frac{1}{2}\hat{\mathbf{R}}\left(\mathbf{B}_{\mathrm{rad}}\cdot\mathbf{B}_{\mathrm{rad}}\right)\right].$$

O sinal negativo global expressa o fato de que $d^2\mathbf{P}$ é o momentum que a carga emite, enquanto que $d\mathbf{P}_m$, que aparece na lei de conservação de momentum linear envolvendo campos e matéria, expressa o ganho de momentum pela carga. Como $\hat{\mathbf{R}}\cdot\mathbf{E}_{\mathrm{rad}}=0$ e $\hat{\mathbf{R}}\cdot\mathbf{B}_{\mathrm{rad}}$, podemos escrever, em analogia com o caso da emissão de energia,

$$\frac{d\mathbf{P}}{dt_{\mathrm{ret}}} = \frac{1}{8\pi}\int_{4\pi}d\Omega\frac{dt}{dt_{\mathrm{ret}}}R^2\hat{\mathbf{R}}\left(\mathbf{E}_{\mathrm{rad}}\cdot\mathbf{E}_{\mathrm{rad}}+\mathbf{B}_{\mathrm{rad}}\cdot\mathbf{B}_{\mathrm{rad}}\right).$$

Como $\mathbf{B}_{\mathrm{rad}}=\hat{\mathbf{R}}\times\mathbf{E}_{\mathrm{rad}}$, segue que

$$\begin{aligned}\mathbf{B}_{\mathrm{rad}}\cdot\mathbf{B}_{\mathrm{rad}} &= \left(\hat{\mathbf{R}}\times\mathbf{E}_{\mathrm{rad}}\right)\cdot\left(\hat{\mathbf{R}}\times\mathbf{E}_{\mathrm{rad}}\right)\\ &= \hat{\mathbf{R}}\cdot\left[\mathbf{E}_{\mathrm{rad}}\times\left(\hat{\mathbf{R}}\times\mathbf{E}_{\mathrm{rad}}\right)\right]\\ &= \hat{\mathbf{R}}\cdot\left[\hat{\mathbf{R}}\left(\mathbf{E}_{\mathrm{rad}}\cdot\mathbf{E}_{\mathrm{rad}}\right)-\mathbf{E}_{\mathrm{rad}}\left(\hat{\mathbf{R}}\cdot\mathbf{E}_{\mathrm{rad}}\right)\right]\end{aligned}$$

e, porque $\mathbf{E}_{\mathrm{rad}}$ é ortogonal a $\hat{\mathbf{R}}$, obtemos

$$\mathbf{B}_{\mathrm{rad}}\cdot\mathbf{B}_{\mathrm{rad}} = \mathbf{E}_{\mathrm{rad}}\cdot\mathbf{E}_{\mathrm{rad}}.$$

Assim,

$$\begin{aligned}\frac{d\mathbf{P}}{dt_{\mathrm{ret}}} &= \frac{1}{4\pi}\int_{4\pi}d\Omega\frac{dt}{dt_{\mathrm{ret}}}R^2\hat{\mathbf{R}}\mathbf{E}_{\mathrm{rad}}\cdot\mathbf{E}_{\mathrm{rad}}\\ &= \frac{1}{4\pi}\int_{4\pi}d\Omega\frac{dt}{dt_{\mathrm{ret}}}R^2\hat{\mathbf{R}}\left|q\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right|^2\\ &= \frac{q^2}{4\pi c^4}\int_{4\pi}d\Omega\frac{dt}{dt_{\mathrm{ret}}}\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^6}.\end{aligned}$$

Da expressão para o tempo retardado,

$$t_{\mathrm{ret}} = t-\frac{\left|\mathbf{r}-\mathbf{r}_0\left(t_{\mathrm{ret}}\right)\right|}{c},$$

segue que

$$\frac{dt}{dt_{\mathrm{ret}}} = 1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}$$

e, portanto,

$$\frac{d\mathbf{P}}{dt_{\text{ret}}} = \frac{q^2}{4\pi c^4}\int_{4\pi} d\Omega\hat{\mathbf{R}}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5}.$$

Sabendo que

$$\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 = -\left(1-\beta^2\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2 + 2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) + a^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2,$$

podemos proceder de maneira análoga ao caso da emissão de energia e calcular a taxa de emissão de momentum linear pela carga puntiforme q. No entanto, aqui vamos utilizar argumentos de covariância para inferir a resposta. Para baixíssimas velocidades comparadas com c, consideremos apenas as contribuições de primeira ordem em $\boldsymbol{\beta}$. Então,

$$\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 \approx -\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2 + 2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}} + a^2\left(1-2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)$$

e

$$\begin{aligned}\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} &\approx \left[-\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2 + 2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}} + a^2\left(1-2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\right]\left(1+5\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\\ &\approx -\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2\left(1+5\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) + 2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}} + a^2\left(1+3\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right).\end{aligned}$$

Integrais sobre a superfície S de um número ímpar de componentes de $\hat{\mathbf{R}}$ são nulas e, portanto,

$$\frac{d\mathbf{P}}{dt_{\text{ret}}} \approx \frac{q^2}{4\pi c^4}\int_{4\pi} d\Omega\hat{\mathbf{R}}\left[-5\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2\hat{\mathbf{R}}\cdot\boldsymbol{\beta} + 2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}} + 3a^2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right].$$

Para calcularmos essa integral, consideremos, digressivamente, o resultado

$$\begin{aligned}\int_{4\pi} d\Omega\frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2} &= \int_0^{2\pi} d\varphi\int_0^{\pi} d\theta\frac{\operatorname{sen}\theta}{\left(1-\beta\cos\theta\right)^2}\\ &= 2\pi\int_{-1}^{1} du\frac{1}{\left(1-\beta u\right)^2}\\ &= \frac{2\pi}{\beta}\left[\frac{1}{(1-\beta)}-\frac{1}{(1+\beta)}\right]\\ &= \frac{4\pi}{(1-\beta^2)}.\end{aligned}$$

Assim, por exemplo,

$$\int_{4\pi} d\Omega\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \frac{1}{6}\frac{\partial^2}{\partial\beta_k\partial\beta_l}\int_{4\pi} d\Omega\frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2}$$

$$= \frac{1}{6}\frac{\partial^2}{\partial\beta_k\partial\beta_l}\left[\frac{4\pi}{(1-\beta^2)}\right].$$

Mas, analogamente ao que fizemos no caso da emissão de energia, podemos calcular as derivadas acima assim:

$$\begin{aligned}
\frac{\partial^2}{\partial\beta_k\partial\beta_l}\left[\frac{4\pi}{(1-\beta^2)}\right] &= 4\pi\frac{\partial}{\partial\beta_k}\left[\frac{2\beta_l}{(1-\beta^2)^2}\right] \\
&= 4\pi\left[\frac{2\delta_{kl}}{(1-\beta^2)^2}+\frac{8\beta_k\beta_l}{(1-\beta^2)^3}\right].
\end{aligned}$$

Logo,

$$\int_{4\pi} d\Omega\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \frac{4\pi}{3}\left[\frac{\delta_{kl}}{(1-\beta^2)^2}+\frac{4\beta_k\beta_l}{(1-\beta^2)^3}\right]$$

e, fazendo $\boldsymbol{\beta}=\mathbf{0}$, obtemos

$$\int_{4\pi} d\Omega\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l = \frac{4\pi}{3}\delta_{kl}.$$

Agora, tomando mais uma derivada da equação acima, antes de tomarmos $\boldsymbol{\beta}=\mathbf{0}$, isto é,

$$\frac{\partial}{\partial\beta_m}\int_{4\pi} d\Omega\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \frac{4\pi}{3}\frac{\partial}{\partial\beta_m}\left[\frac{\delta_{kl}}{(1-\beta^2)^2}+\frac{4\beta_k\beta_l}{(1-\beta^2)^3}\right],$$

obtemos

$$\begin{aligned}
\int_{4\pi} d\Omega\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_m}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} &= \frac{\pi}{3}\left\{\frac{4\beta_m\delta_{kl}}{(1-\beta^2)^3}+\frac{\partial}{\partial\beta_m}\left[\frac{4\beta_k\beta_l}{(1-\beta^2)^3}\right]\right\} \\
&= \frac{4\pi}{3}\left[\frac{\beta_m\delta_{kl}+\beta_l\delta_{km}+\beta_k\delta_{lm}}{(1-\beta^2)^3}+\frac{6\beta_k\beta_l\beta_m}{(1-\beta^2)^4}\right],
\end{aligned}$$

implicando, quando escolhemos $\boldsymbol{\beta}=\mathbf{0}$, que

$$\int_{4\pi} d\Omega\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l = 0,$$

conforme adiantamos acima. É fácil agora tomarmos mais uma derivada da equação acima, antes de tomarmos $\boldsymbol{\beta}=\mathbf{0}$, isto é,

$$\frac{\partial}{\partial\beta_n}\int_{4\pi} d\Omega\frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_m}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} = \frac{4\pi}{3}\frac{\partial}{\partial\beta_n}\left[\frac{\beta_m\delta_{kl}+\beta_l\delta_{km}+\beta_k\delta_{lm}}{(1-\beta^2)^3}+\frac{6\beta_k\beta_l\beta_m}{(1-\beta^2)^4}\right],$$

ou seja,

$$\begin{aligned}
\int_{4\pi} d\Omega \frac{\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_m\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_n}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^6} &= \frac{4\pi}{15}\frac{\partial}{\partial\beta_n}\left[\frac{\beta_m\delta_{kl}+\beta_l\delta_{km}+\beta_k\delta_{lm}}{(1-\beta^2)^3}+\frac{6\beta_k\beta_l\beta_m}{(1-\beta^2)^4}\right] \\
&= \frac{4\pi}{15}\frac{\delta_{mn}\delta_{kl}+\delta_{ln}\delta_{km}+\delta_{kn}\delta_{lm}}{(1-\beta^2)^3} \\
&+ \frac{8\pi}{15}\frac{\beta_n\beta_m\delta_{kl}+\beta_n\beta_l\delta_{km}+\beta_n\beta_k\delta_{lm}}{(1-\beta^2)^4} \\
&+ \frac{24\pi}{15}\frac{\delta_{kn}\beta_l\beta_m+\beta_k\delta_{ln}\beta_m+\beta_k\beta_l\delta_{mn}}{(1-\beta^2)^4}+\frac{192\pi}{3}\frac{\beta_k\beta_l\beta_m\beta_n}{(1-\beta^2)^5}.
\end{aligned}$$

Assim, com $\boldsymbol{\beta}=\mathbf{0}$, obtemos

$$\int_{4\pi} d\Omega\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_m\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_n = \frac{4\pi}{15}\left(\delta_{mn}\delta_{kl}+\delta_{ln}\delta_{km}+\delta_{kn}\delta_{lm}\right).$$

Com essa digressão sobre integrais angulares de componentes da normal à superfície esférica S, obtemos

$$\begin{aligned}
\int_{4\pi} d\Omega\hat{\mathbf{R}}\left[-5\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right] &= -5\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\hat{\mathbf{x}}_k a_l a_m\beta_n\int_{4\pi} d\Omega\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_m\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_n \\
&= -\frac{4\pi}{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\hat{\mathbf{x}}_k a_l a_m\beta_n\left(\delta_{mn}\delta_{kl}+\delta_{ln}\delta_{km}+\delta_{kn}\delta_{lm}\right) \\
&= -\frac{4\pi}{3}\left(2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}+\boldsymbol{\beta}a^2\right),
\end{aligned}$$

$$\begin{aligned}
\int_{4\pi} d\Omega\hat{\mathbf{R}}\left(2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}}\right) &= 2\mathbf{a}\cdot\boldsymbol{\beta}\sum_{k=1}^{3}\sum_{l=1}^{3}\hat{\mathbf{x}}_k a_l\int_{4\pi} d\Omega\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l \\
&= \frac{8\pi}{3}\mathbf{a}\cdot\boldsymbol{\beta}\sum_{k=1}^{3}\sum_{l=1}^{3}\hat{\mathbf{x}}_k a_l\delta_{kl} \\
&= \frac{8\pi}{3}\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}
\end{aligned}$$

e

$$\begin{aligned}
\int_{4\pi} d\Omega\hat{\mathbf{R}}\left(3a^2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right) &= 3a^2\sum_{k=1}^{3}\sum_{l=1}^{3}\hat{\mathbf{x}}_k\beta_l\int_{4\pi} d\Omega\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_k\hat{\mathbf{R}}\cdot\hat{\mathbf{x}}_l \\
&= 4\pi a^2\sum_{k=1}^{3}\sum_{l=1}^{3}\hat{\mathbf{x}}_k\beta_l\delta_{kl} \\
&= 4\pi a^2\boldsymbol{\beta}.
\end{aligned}$$

Portanto, até primeira ordem em $\boldsymbol{\beta}$, podemos escrever

$$\frac{d\mathbf{P}}{dt_{\rm ret}} \approx \frac{q^2}{4\pi c^4}\left[-\frac{4\pi}{3}\left(2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}+\boldsymbol{\beta}a^2\right)+\frac{8\pi}{3}\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}+4\pi a^2\boldsymbol{\beta}\right],$$

isto é,

$$\frac{d\mathbf{P}}{dt_{\rm ret}} \approx \frac{2q^2}{3c^4}\boldsymbol{\beta}a^2.$$

Agora que temos a taxa de emissão de momentum para baixas velocidades, podemos procurar por uma expressão covariante desse resultado. Ao invés de $d\mathbf{P}/dt_{\rm ret}$, consideremos $d\mathbf{P}/d\tau$, onde τ é o tempo próprio da partícula no instante $t_{\rm ret}$. Como

$$\begin{aligned}\frac{d\mathbf{P}}{d\tau} &= \frac{dt_{\rm ret}}{d\tau}\frac{d\mathbf{P}}{dt_{\rm ret}}\\ &= \frac{1}{\sqrt{1-\beta^2}}\frac{d\mathbf{P}}{dt_{\rm ret}}\end{aligned}$$

e, portanto,

$$\frac{d\mathbf{P}}{d\tau} \approx \frac{d\mathbf{P}}{dt_{\rm ret}}$$

até primeira ordem em $\boldsymbol{\beta}$, podemos escrever, nessa ordem,

$$\frac{d\mathbf{P}}{d\tau} \approx \frac{2q^2}{3c^4}\boldsymbol{\beta}a^2.$$

Também é um fato que

$$\begin{aligned}\mathbf{u} &= \frac{d\mathbf{r}_q\left(t_{\rm ret}\right)}{d\tau}\\ &= \frac{dt_{\rm ret}}{d\tau}\frac{d\mathbf{r}_q\left(t_{\rm ret}\right)}{dt_{\rm ret}}\\ &= \frac{c}{\sqrt{1-\beta^2}}\boldsymbol{\beta}\end{aligned}$$

e, assim, até primeira ordem em $\boldsymbol{\beta}$, vale a aproximação

$$\mathbf{u} \approx c\boldsymbol{\beta}.$$

Logo, nessa ordem de aproximação,

$$\frac{d\mathbf{P}}{d\tau} \approx \frac{2q^2}{3c^5}\mathbf{u}a^2.$$

Podemos definir também uma aceleração em termos do tempo próprio como

$$\begin{aligned}\boldsymbol{\alpha} &= \frac{d\mathbf{u}}{d\tau}\\ &= \frac{dt_{\rm ret}}{d\tau}\frac{d\mathbf{u}}{dt_{\rm ret}}\\ &= \frac{1}{\sqrt{1-\beta^2}}\frac{d\mathbf{u}}{dt_{\rm ret}}.\end{aligned}$$

Se definirmos também a componente do tipo temporal:

$$\begin{aligned}\alpha^0 &= \frac{du^0}{d\tau} \\ &= \frac{d^2\left(ct_{\mathrm{ret}}\right)}{d\tau^2} \\ &= \frac{1}{\sqrt{1-\beta^2}}\frac{d}{dt_{\mathrm{ret}}}\left(\frac{c}{\sqrt{1-\beta^2}}\right),\end{aligned}$$

isto é,

$$\begin{aligned}\alpha^0 &= \frac{c}{2}\frac{d}{dt_{\mathrm{ret}}}\left(\frac{1}{1-\beta^2}\right) \\ &= \frac{\boldsymbol{\beta}\cdot\mathbf{a}}{\left(1-\beta^2\right)^2},\end{aligned}$$

vemos que $\left(\alpha^0,\boldsymbol{\alpha}\right) = d\left(u^0,\mathbf{u}\right)/d\tau$ forma um quadrivetor, já que $\left(u^0,\mathbf{u}\right)$ é um quadrivetor e $d\tau$ é um invariante. Notemos que

$$\alpha^\nu\alpha_\nu = \frac{\left(\boldsymbol{\beta}\cdot\mathbf{a}\right)^2}{\left(1-\beta^2\right)^4} - \frac{1}{1-\beta^2}\left(\frac{d\mathbf{u}}{dt_{\mathrm{ret}}}\right)^2.$$

Mas,

$$\begin{aligned}\frac{d\mathbf{u}}{dt_{\mathrm{ret}}} &= \frac{d}{dt_{\mathrm{ret}}}\left(\frac{c}{\sqrt{1-\beta^2}}\boldsymbol{\beta}\right) \\ &= \boldsymbol{\beta}\frac{d}{dt_{\mathrm{ret}}}\left(\frac{c}{\sqrt{1-\beta^2}}\right) + \frac{c}{\sqrt{1-\beta^2}}\frac{d\boldsymbol{\beta}}{dt_{\mathrm{ret}}} \\ &= \frac{1}{\sqrt{1-\beta^2}}\left[\mathbf{a} + \frac{\boldsymbol{\beta}\cdot\mathbf{a}\boldsymbol{\beta}}{\left(1-\beta^2\right)}\right]\end{aligned}$$

e, com isso,

$$\begin{aligned}\alpha^\nu\alpha_\nu &= \frac{\left(\boldsymbol{\beta}\cdot\mathbf{a}\right)^2}{\left(1-\beta^2\right)^4} - \frac{1}{\left(1-\beta^2\right)^2}\left[\mathbf{a} + \frac{\boldsymbol{\beta}\cdot\mathbf{a}\boldsymbol{\beta}}{\left(1-\beta^2\right)}\right]^2 \\ &= \frac{\left(\boldsymbol{\beta}\cdot\mathbf{a}\right)^2}{\left(1-\beta^2\right)^4} - \frac{a^2}{\left(1-\beta^2\right)^2} - 2\frac{\left(\boldsymbol{\beta}\cdot\mathbf{a}\right)^2}{\left(1-\beta^2\right)^3} - \frac{\left(\boldsymbol{\beta}\cdot\mathbf{a}\right)^2}{\left(1-\beta^2\right)^4}\beta^2 \\ &= -\frac{\left(\boldsymbol{\beta}\cdot\mathbf{a}\right)^2}{\left(1-\beta^2\right)^3} - \frac{a^2}{\left(1-\beta^2\right)^2},\end{aligned}$$

isto é,

$$\begin{aligned}\alpha^\nu\alpha_\nu &= -\frac{a^2 - a^2\beta^2 + \left(\boldsymbol{\beta}\cdot\mathbf{a}\right)^2}{\left(1-\beta^2\right)^3} \\ &= -\frac{a^2 - \left(\mathbf{a}\times\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^3}.\end{aligned}$$

Até primeira ordem em $\boldsymbol{\beta}$, portanto,

$$\alpha^{\nu}\alpha_{\nu} \quad \approx \quad -a^2$$

e podemos escrever, nessa ordem,

$$\frac{d\mathbf{P}}{d\tau} \quad \approx \quad -\frac{2q^2}{3c^5}\mathbf{u}\alpha^{\nu}\alpha_{\nu}.$$

No caso da emissão de energia, vimos que a taxa de emissão de energia é dada por

$$\frac{dW}{dt_{\text{ret}}} \quad = \quad \frac{2q^2}{3c^3}\left[\frac{a^2-(\mathbf{a}\times\boldsymbol{\beta})^2}{(1-\beta^2)^3}\right],$$

ou seja, à luz de nossos cálculos acima para $\alpha^{\nu}\alpha_{\nu}$, podemos escrever

$$\frac{dW}{dt_{\text{ret}}} \quad = \quad -\frac{2q^2}{3c^3}\alpha^{\nu}\alpha_{\nu},$$

ou ainda,

$$\begin{aligned}\frac{1}{c}\frac{dW}{d\tau} \quad &= \quad -\frac{2q^2}{3c^4}\frac{dt_{\text{ret}}}{d\tau}\alpha^{\nu}\alpha_{\nu}\\ &= \quad -\frac{2q^2}{3c^5}\frac{d\left(ct_{\text{ret}}\right)}{d\tau}\alpha^{\nu}\alpha_{\nu}\\ &= \quad -\frac{2q^2}{3c^5}u^0\alpha^{\nu}\alpha_{\nu}.\end{aligned}$$

Essa expressão é exata, valendo para qualquer valor de $\boldsymbol{\beta}$. A taxa de energia emitida, dividida por c, é a componente temporal de um quadrivetor, como esperado. Com esse resultado e a equação

$$\frac{d\mathbf{P}}{d\tau} \quad \approx \quad -\frac{2q^2}{3c^5}\mathbf{u}\alpha^{\nu}\alpha_{\nu}$$

vemos que

$$\frac{d}{d\tau}\left(\frac{W}{c},\mathbf{P}\right) \quad \approx \quad -\frac{2q^2}{3c^5}\alpha^{\nu}\alpha_{\nu}\left(u^0,\mathbf{u}\right),$$

que é, por si só, uma equação covariante e, portanto, deve valer em qualquer sistema de referência, inclusive com $\beta \lesssim 1$. Podemos inferir, portanto, que, ao invés de uma aproximação, temos

$$\frac{d}{d\tau}\left(\frac{W}{c},\mathbf{P}\right) \quad = \quad -\frac{2q^2}{3c^5}\alpha^{\nu}\alpha_{\nu}\left(u^0,\mathbf{u}\right),$$

exatamente e, portanto, a emissão de momentum pode ser inferida como sendo dada pela parte espacial dessa equação, isto é,

$$\frac{d\mathbf{P}}{dt_{\text{ret}}}\frac{dt_{\text{ret}}}{d\tau} \quad = \quad -\frac{2q^2}{3c^5}\alpha^{\nu}\alpha_{\nu}\mathbf{u},$$

ou seja,

$$\frac{d\mathbf{P}}{dt_{\mathrm{ret}}} = \frac{2q^2}{3c^4}\left[\frac{a^2-(\mathbf{a}\times\boldsymbol{\beta})^2}{(1-\beta^2)^3}\right]\boldsymbol{\beta}.$$

Se seguirmos ao longo das linhas que seguimos no caso da emissão de energia, encontraremos exatamente esse resultado, sem ter que inferi-lo. O cálculo direto, além de direto, é mais simples do que o apresentado aqui. No entanto, a presente inferência ilustra como normalmente podemos obter equações covariantes através da generalização de suas contra-partes válidas para velocidades não relativísticas.