Emissão de energia por uma partícula carregada em movimento arbitrário

Vamos calcular a potência irradiada por uma partícula em movimento arbitrário.
Os campos de radiação, $\mathbf{E}_{\text{rad}}$ e $\mathbf{B}_{\text{rad}}$, produzidos por uma carga q que descreve uma trajetória $\mathbf{r}_0\left(t\right)$ arbitrária são deduzidos a partir dos potenciais de Liénard-Wiechert:

$$\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right) = q\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}$$

e

$$\mathbf{B}_{\text{rad}}\left(\mathbf{r},t\right) = \hat{\mathbf{R}}\times\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right),$$

onde

$$\mathbf{R} = \mathbf{r}-\mathbf{r}_0\left(t_{\text{ret}}\right),$$

$$t_{\text{ret}} = t-\frac{\left|\mathbf{r}-\mathbf{r}_0\left(t_{\text{ret}}\right)\right|}{c},$$

$$R = \left|\mathbf{R}\right|,$$

$$\hat{\mathbf{R}} = \frac{\mathbf{R}}{R},$$

$$\begin{aligned}\mathbf{v} &= \mathbf{v}\left(t_{\text{ret}}\right)\\ &= \frac{d\mathbf{r}_0\left(t_{\text{ret}}\right)}{dt_{\text{ret}}}\\ &= \dot{\mathbf{r}}_0\left(t_{\text{ret}}\right)\end{aligned}$$

e

$$\boldsymbol{\beta} = \frac{\mathbf{v}}{c}.$$

É importante notarmos que os campos são dados para o ponto arbitrário de observação, $\mathbf{r}$, no instante arbitrário de observação, t. Nesse ponto do espaço-tempo, entretanto, a radiação que ali se encontre, necessariamente terá sido a emitida pela partícula no instante retardado t_{ret} e na posição $\mathbf{r}_0\left(t_{\text{ret}}\right)$. O vetor de Poynting de radiação, nesse caso, fica

$$\begin{aligned}\mathbf{S}_{\text{rad}}\left(\mathbf{r},t\right) &= \frac{c}{4\pi}\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right)\times\mathbf{B}_{\text{rad}}\left(\mathbf{r},t\right)\\ &= \frac{c}{4\pi}\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right)\times\left[\hat{\mathbf{R}}\times\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right)\right]\\ &= \frac{c}{4\pi}\hat{\mathbf{R}}\left[\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right)\cdot\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right)\right]-\frac{c}{4\pi}\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right)\hat{\mathbf{R}}\cdot\mathbf{E}_{\text{rad}}\left(\mathbf{r},t\right).\end{aligned}$$

Como

$$\hat{\mathbf{R}} \cdot \mathbf{E}_{\text{rad}}(\mathbf{r},t) = \mathbf{0},$$

segue que

$$\mathbf{S}_{\text{rad}}(\mathbf{r},t) = \frac{c}{4\pi}\hat{\mathbf{R}}\left|\mathbf{E}_{\text{rad}}(\mathbf{r},t)\right|^2.$$

Seja $d^2W/d\Omega$ a energia por unidade de ângulo sólido $d\Omega$ emitida pela partícula durante o intervalo de tempo dt_{ret}. Seja S a superfície esférica, centrada em $\mathbf{r}_0(t_{\text{ret}})$, com raio R. Dentro do ângulo sólido $d\Omega$, em torno do vetor $\mathbf{R}$, a energia d^2W leva um intervalo de tempo dt para passar através do elemento de área $R^2 d\Omega$. Podemos escrever, portanto,

$$\frac{d^2W}{d\Omega dt} = R^2\hat{\mathbf{R}} \cdot \mathbf{S}_{\text{rad}}(\mathbf{r},t),$$

isto é,

$$d^2W = dt d\Omega R^2\hat{\mathbf{R}} \cdot \mathbf{S}_{\text{rad}}(\mathbf{r},t).$$

Como essa mesma quantidade de energia é a que a partícula emite em um intervalo dt_{ret}, segue que a potência emitida pela partícula, no ponto $\mathbf{r}_0(t_{\text{ret}})$ de sua trajetória, é dada por

$$\frac{dW}{dt_{\text{ret}}} = \int_{4\pi} d\Omega \frac{dt}{dt_{\text{ret}}} R^2\hat{\mathbf{R}} \cdot \mathbf{S}_{\text{rad}}(\mathbf{r},t),$$

onde a integral é sobre todo o ângulo sólido compreendido por S. Como

$$t_{\text{ret}} = t - \frac{\left|\mathbf{r}-\mathbf{r}_0(t_{\text{ret}})\right|}{c},$$

segue que

$$t = t_{\text{ret}} + \frac{\left|\mathbf{r}-\mathbf{r}_0(t_{\text{ret}})\right|}{c}$$

e, portanto,

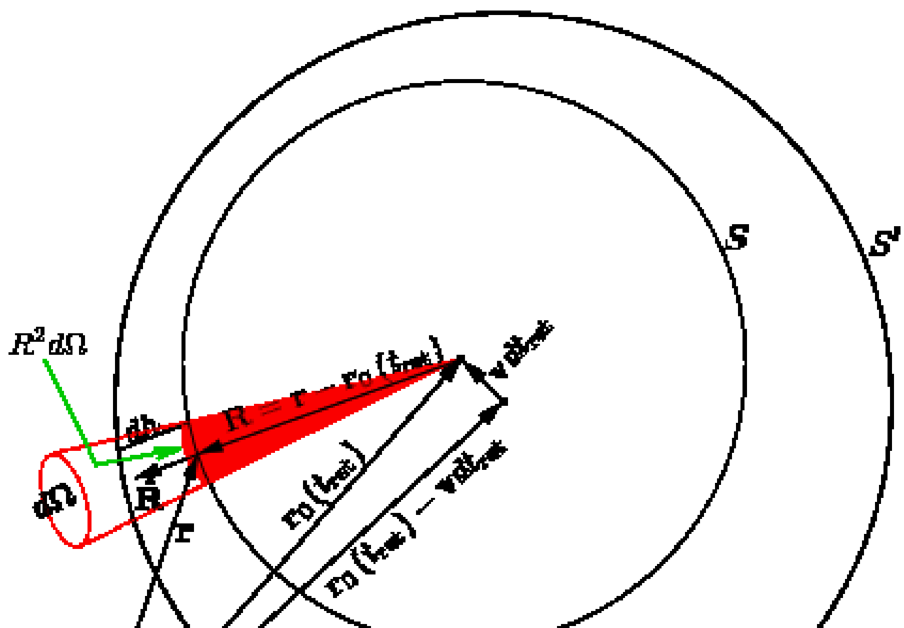
$$\begin{aligned}\frac{dt}{dt_{\text{ret}}} &= 1 - \left[\frac{\mathbf{r}-\mathbf{r}_0(t_{\text{ret}})}{c\left|\mathbf{r}-\mathbf{r}_0(t_{\text{ret}})\right|}\right] \cdot \frac{d\mathbf{r}_0(t_{\text{ret}})}{dt_{\text{ret}}} \\ &= 1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}.\end{aligned}$$

Com isso, vemos que a potência irradiada pela partícula é dada por

$$\begin{aligned}\frac{dW}{dt_{\text{ret}}} &= \frac{c}{4\pi}\int_{4\pi} d\Omega \left(1-\hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right) R^2\hat{\mathbf{R}} \cdot \hat{\mathbf{R}}\left|\mathbf{E}_{\text{rad}}(\mathbf{r},t)\right|^2 \\ &= \frac{c}{4\pi}\int_{4\pi} d\Omega \left(1-\hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right) R^2\hat{\mathbf{R}} \cdot \hat{\mathbf{R}}\left|q\frac{\hat{\mathbf{R}} \times \left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right) \times \mathbf{a}\right]}{Rc^2\left(1-\hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right)^3}\right|^2,\end{aligned}$$

isto é,

$$
\begin{aligned}
\frac{dW}{dt_{\mathrm{ret}}} &= \frac{q^2}{4\pi c^3}\int_{4\pi} d\Omega\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\left|\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right|^2, \\
&= \frac{q^2}{4\pi c^3}\int_{4\pi} d\Omega\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5}.
\end{aligned}
$$



Outra maneira de deduzir essa mesma expressão para a potência irradiada é a seguinte. Considere que a superfície S esteja centrada no ponto $\mathbf{r}_0\left(t_{\mathrm{ret}}\right)$ e que o cálculo que estamos fazendo seja no instante t. Durante um intervalo de tempo dt_{ret}, antes de t_{ret}, a partícula estava em $\mathbf{r}_0(t_{\text{ret}}) - \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}$. A radiação se propaga com velocidade de magnitude c. Então, no instante $t > t_{\text{ret}}$, toda a energia emitida pela partícula entre os instantes $t_{\text{ret}} - dt_{\text{ret}}$ e t_{ret} se encontra entre duas superfícies esféricas de raios $c(t - t_{\text{ret}})$ e $c(t - t_{\text{ret}} + dt_{\text{ret}})$, a primeira, S, centrada em $\mathbf{r}_0(t_{\text{ret}})$ e a segunda, S', centrada em $\mathbf{r}_0(t_{\text{ret}}) - \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}$. Note que a segunda superfície esférica, S', contém a primeira, S, pois teve mais tempo para se expandir com velocidade c. Um ângulo sólido $d\Omega$, a partir de $\mathbf{r}_0(t_{\text{ret}})$, define, sobre S, um elemento de área de normal dada por $\hat{\mathbf{R}}$ e magnitude $R^2 d\Omega$. O elemento de volume entre esse elemento de área de S e a superfície esférica S', de raio $c(t - t_{\text{ret}} + dt_{\text{ret}})$ e centrada em $\mathbf{r}_0(t_{\text{ret}}) - \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}$, contém a energia que a partícula irradiou, durante dt_{ret}, no ângulo sólido $d\Omega$. Seja $\mathbf{r}$ o ponto de S onde calculamos a normal $\hat{\mathbf{R}}$. Então, sendo $\hat{\mathbf{R}}dh$ o vetor que vai de $\mathbf{r}$ até a superfície esférica S', seguem as relações:

$$\begin{aligned} c(t - t_{\text{ret}}) &= |\mathbf{r} - \mathbf{r}_0(t_{\text{ret}})| \\ &= R \end{aligned}$$

e

$$\begin{aligned} c(t - t_{\text{ret}} + dt_{\text{ret}}) &= \left|\mathbf{r} + \hat{\mathbf{R}}dh - [\mathbf{r}_0(t_{\text{ret}}) - \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}]\right| \\ &= \left|\mathbf{r} - \mathbf{r}_0(t_{\text{ret}}) + \hat{\mathbf{R}}dh + \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}\right| \\ &= \left|R\hat{\mathbf{R}} + dh\hat{\mathbf{R}} + \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}\right|. \end{aligned}$$

Como $R \gg dh, |\mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}|$, segue que

$$\begin{aligned} \left|R\hat{\mathbf{R}} + dh\hat{\mathbf{R}} + \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}\right| &= R\left|\hat{\mathbf{R}} + \frac{dh}{R}\hat{\mathbf{R}} + \frac{\mathbf{v}(t_{\text{ret}})}{R}dt_{\text{ret}}\right| \\ &\approx R\sqrt{1 + 2\frac{dh}{R} + 2\frac{\hat{\mathbf{R}} \cdot \mathbf{v}(t_{\text{ret}})}{R}dt_{\text{ret}}} \\ &\approx R + dh + \hat{\mathbf{R}} \cdot \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}. \end{aligned}$$

Assim,

$$c(t - t_{\text{ret}} + dt_{\text{ret}}) \approx R + dh + \hat{\mathbf{R}} \cdot \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}}$$

e, como $c(t - t_{\text{ret}}) = R$, vem

$$c\,dt_{\text{ret}} \approx dh + \hat{\mathbf{R}} \cdot \mathbf{v}(t_{\text{ret}})\,dt_{\text{ret}},$$

isto é,

$$dh \approx c\left(1 - \hat{\mathbf{R}} \cdot \boldsymbol{\beta}\right) dt_{\text{ret}}.$$

A densidade de energia no ponto $\mathbf{r}$ é dada por

$$u_{\rm rad} = \frac{1}{8\pi}\left(\mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad}+\mathbf{B}_{\rm rad}\cdot\mathbf{B}_{\rm rad}\right).$$

Como $\mathbf{B}_{\rm rad}=\hat{\mathbf{R}}\times\mathbf{E}_{\rm rad}$, segue que

$$\begin{aligned}\mathbf{B}_{\rm rad}\cdot\mathbf{B}_{\rm rad} &= \left(\hat{\mathbf{R}}\times\mathbf{E}_{\rm rad}\right)\cdot\left(\hat{\mathbf{R}}\times\mathbf{E}_{\rm rad}\right)\\ &= \hat{\mathbf{R}}\cdot\left[\mathbf{E}_{\rm rad}\times\left(\hat{\mathbf{R}}\times\mathbf{E}_{\rm rad}\right)\right]\\ &= \hat{\mathbf{R}}\cdot\left[\hat{\mathbf{R}}\left(\mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad}\right)-\mathbf{E}_{\rm rad}\left(\hat{\mathbf{R}}\cdot\mathbf{E}_{\rm rad}\right)\right]\end{aligned}$$

e, porque $\mathbf{E}_{\rm rad}$ é ortogonal a $\hat{\mathbf{R}}$, obtemos

$$\mathbf{B}_{\rm rad}\cdot\mathbf{B}_{\rm rad} = \mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad}.$$

Logo,

$$u_{\rm rad} = \frac{1}{4\pi}\mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad}.$$

A energia irradiada pela partícula durante o intervalo $dt_{\rm ret}$ e dentro do ângulo sólido $d\Omega$ é, portanto,

$$\begin{aligned}d^2W &= u_{\rm rad}\,dh R^2 d\Omega\\ &= d\Omega\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)c u_{\rm rad}R^2 dt_{\rm ret}.\end{aligned}$$

Com isso,

$$\begin{aligned}\frac{dW}{dt_{\rm ret}} &= \int_{4\pi}d\Omega\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)c\frac{1}{4\pi}\mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad}R^2\\ &= \int_{4\pi}d\Omega\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)c\frac{1}{4\pi}\left|q\frac{\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]}{Rc^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}\right|^2 R^2,\end{aligned}$$

isto é,

$$\frac{dW}{dt_{\rm ret}} = \frac{q^2}{4\pi c^3}\int_{4\pi}d\Omega\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5},$$

como anteriormente.

O numerador do integrando pode ser escrito como

$$\begin{aligned}\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 &= \left|\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\hat{\mathbf{R}}\cdot\mathbf{a}-\mathbf{a}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\right|^2\\ &= \left|\hat{\mathbf{R}}-\boldsymbol{\beta}\right|^2\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2-2\mathbf{a}\cdot\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\hat{\mathbf{R}}\cdot\mathbf{a}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\\ &+ \mathbf{a}\cdot\mathbf{a}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2,\end{aligned}$$

isto é,

$$\begin{aligned}\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 &= \left(1-2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}+\beta^2\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2\\ &- \left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2\left(2-2\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)+2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\\ &+ a^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2,\end{aligned}$$

ou seja,

$$\begin{aligned}\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2 &= -\left(1-\beta^2\right)\left(\hat{\mathbf{R}}\cdot\mathbf{a}\right)^2\\ &+ 2\mathbf{a}\cdot\boldsymbol{\beta}\mathbf{a}\cdot\hat{\mathbf{R}}\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)\\ &+ a^2\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^2.\end{aligned}$$

Assim,

$$\begin{aligned}\frac{dW}{dt_{\text{ret}}} &= \frac{q^2}{4\pi c^3}\int_{4\pi}d\Omega\frac{\left|\hat{\mathbf{R}}\times\left[\left(\hat{\mathbf{R}}-\boldsymbol{\beta}\right)\times\mathbf{a}\right]\right|^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5}\\ &= -\frac{q^2\left(1-\beta^2\right)}{4\pi c^3}\int_{4\pi}d\Omega\frac{\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5}\\ &+ \frac{2q^2\mathbf{a}\cdot\boldsymbol{\beta}}{4\pi c^3}\int_{4\pi}d\Omega\frac{\mathbf{a}\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4}\\ &+ \frac{q^2a^2}{4\pi c^3}\int_{4\pi}d\Omega\frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}.\end{aligned}$$

As integrais acima são sobre os ângulos que definem a direção e o sentido do versor $\hat{\mathbf{R}}$, enquanto $\boldsymbol{\beta}$ é um vetor fixo na integração. Podemos, portanto, escolher o eixo z ao longo de $\boldsymbol{\beta}$ e integrar sobre os ângulos θ e φ. Logo,

$$\begin{aligned}\int_{4\pi}d\Omega\frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3} &= \int_0^{2\pi}d\varphi\int_0^{\pi}d\theta\frac{\text{sen}\theta}{\left(1-\beta\cos\theta\right)^3}\\ &= 2\pi\int_{-1}^{1}du\frac{1}{\left(1-\beta u\right)^3}\\ &= \frac{\pi}{\beta}\left[\frac{1}{\left(1-\beta\right)^2}-\frac{1}{\left(1+\beta\right)^2}\right]\\ &= \frac{4\pi}{\left(1-\beta^2\right)^2}.\end{aligned}$$

Também podemos escrever

$$\int_{4\pi} d\Omega \frac{\mathbf{a}\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \sum_{k=1}^{3} \mathbf{a}\cdot\hat{\mathbf{x}}_k \int_{4\pi} d\Omega \frac{\hat{\mathbf{x}}_k\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4}$$

$$= \sum_{k=1}^{3} \frac{\mathbf{a}\cdot\hat{\mathbf{x}}_k}{3} \frac{\partial}{\partial\beta_k} \int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3},$$

que, usando o resultado da integral calculada logo acima, resulta em

$$\int_{4\pi} d\Omega \frac{\mathbf{a}\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \sum_{k=1}^{3} \frac{4\pi}{3}\mathbf{a}\cdot\hat{\mathbf{x}}_k \frac{\partial}{\partial\beta_k}\left[\frac{1}{(1-\beta^2)^2}\right]$$

$$= \sum_{k=1}^{3} \frac{4\pi}{3}\mathbf{a}\cdot\hat{\mathbf{x}}_k \left[\frac{2}{(1-\beta^2)^3}\right]\frac{\partial\beta^2}{\partial\beta_k}$$

$$= \sum_{k=1}^{3} \frac{4\pi}{3}\mathbf{a}\cdot\hat{\mathbf{x}}_k \left[\frac{2}{(1-\beta^2)^3}\right]2\beta_k,$$

isto é,

$$\int_{4\pi} d\Omega \frac{\mathbf{a}\cdot\hat{\mathbf{R}}}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^4} = \frac{16\pi\mathbf{a}\cdot\boldsymbol{\beta}}{3\,(1-\beta^2)^3}.$$

Finalmente, a integral restante pode ser calculada de forma análoga e obtemos

$$\int_{4\pi} d\Omega \frac{\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} = \sum_{k=1}^{3}\sum_{l=1}^{3} \mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l \int_{4\pi} d\Omega \frac{\left(\hat{\mathbf{x}}_k\cdot\hat{\mathbf{R}}\right)\left(\hat{\mathbf{x}}_l\cdot\hat{\mathbf{R}}\right)}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5}$$

$$= \sum_{k=1}^{3}\sum_{l=1}^{3} \frac{\mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l}{12} \frac{\partial}{\partial\beta_k}\frac{\partial}{\partial\beta_l} \int_{4\pi} d\Omega \frac{1}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^3}$$

$$= \sum_{k=1}^{3}\sum_{l=1}^{3} \frac{\pi\mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l}{3} \frac{\partial}{\partial\beta_k}\frac{\partial}{\partial\beta_l}\left[\frac{1}{(1-\beta^2)^2}\right].$$

Mas,

$$\frac{\partial}{\partial\beta_k}\frac{\partial}{\partial\beta_l}\left[\frac{1}{(1-\beta^2)^2}\right] = \frac{\partial}{\partial\beta_k}\left\{2\beta_l\frac{\partial}{\partial\beta^2}\left[\frac{1}{(1-\beta^2)^2}\right]\right\}$$

$$= \frac{\partial}{\partial\beta_k}\left[\frac{4\beta_l}{(1-\beta^2)^3}\right]$$

$$= \frac{4\delta_{kl}}{(1-\beta^2)^3} + 4\beta_l\frac{\partial}{\partial\beta_k}\left[\frac{1}{(1-\beta^2)^3}\right]$$

e, portanto,

$$\begin{aligned}\frac{\partial}{\partial\beta_k}\frac{\partial}{\partial\beta_l}\left[\frac{1}{\left(1-\beta^2\right)^2}\right] &= \frac{4\delta_{kl}}{\left(1-\beta^2\right)^3}+8\beta_k\beta_l\frac{\partial}{\partial\beta^2}\left[\frac{1}{\left(1-\beta^2\right)^3}\right]\\ &= \frac{4\delta_{kl}}{\left(1-\beta^2\right)^3}+\frac{24\beta_k\beta_l}{\left(1-\beta^2\right)^4}.\end{aligned}$$

Com isso,

$$\begin{aligned}\int_{4\pi} d\Omega\frac{\left(\mathbf{a}\cdot\hat{\mathbf{R}}\right)^2}{\left(1-\hat{\mathbf{R}}\cdot\boldsymbol{\beta}\right)^5} &= \sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\pi\mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l}{3}\frac{4\delta_{kl}}{\left(1-\beta^2\right)^3}\\ &+ \sum_{k=1}^{3}\sum_{l=1}^{3}\frac{\pi\mathbf{a}\cdot\hat{\mathbf{x}}_k\mathbf{a}\cdot\hat{\mathbf{x}}_l}{3}\frac{24\beta_k\beta_l}{\left(1-\beta^2\right)^4}\\ &= \frac{4\pi a^2}{3\left(1-\beta^2\right)^3}+\frac{8\pi\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^4}.\end{aligned}$$

Utilizando esses resultados, concluímos que a potência irradiada pela partícula carregada é dada por

$$\begin{aligned}\frac{dW}{dt_{\text{ret}}} &= -\frac{q^2\left(1-\beta^2\right)}{4\pi c^3}\frac{4\pi a^2}{3\left(1-\beta^2\right)^3}-\frac{q^2\left(1-\beta^2\right)}{4\pi c^3}\frac{8\pi\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^4}\\ &+ \frac{2q^2\mathbf{a}\cdot\boldsymbol{\beta}}{4\pi c^3}\frac{16\pi\mathbf{a}\cdot\boldsymbol{\beta}}{3\left(1-\beta^2\right)^3}\\ &+ \frac{q^2a^2}{4\pi c^3}\frac{4\pi}{\left(1-\beta^2\right)^2},\end{aligned}$$

isto é,

$$\begin{aligned}\frac{dW}{dt_{\text{ret}}} &= \frac{2q^2}{3c^3}\left[\frac{\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^3}+\frac{a^2}{\left(1-\beta^2\right)^2}\right]\\ &= \frac{2q^2}{3c^3}\left[\frac{\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2+a^2-a^2\beta^2}{\left(1-\beta^2\right)^3}\right].\end{aligned}$$

Podemos simplificar ainda mais essa expressão observando que

$$\begin{aligned}\left(\mathbf{a}\times\boldsymbol{\beta}\right)^2 &= \left(\mathbf{a}\times\boldsymbol{\beta}\right)\cdot\left(\mathbf{a}\times\boldsymbol{\beta}\right)\\ &= \mathbf{a}\cdot\left[\boldsymbol{\beta}\times\left(\mathbf{a}\times\boldsymbol{\beta}\right)\right]\\ &= \mathbf{a}\cdot\left[\mathbf{a}\beta^2-\boldsymbol{\beta}\mathbf{a}\cdot\boldsymbol{\beta}\right]\\ &= a^2\beta^2-\left(\mathbf{a}\cdot\boldsymbol{\beta}\right)^2\end{aligned}$$

e, portanto,

$$\frac{dW}{dt_{\text{ret}}} = \frac{2q^2}{3c^3}\left[\frac{a^2-\left(\mathbf{a}\times\boldsymbol{\beta}\right)^2}{\left(1-\beta^2\right)^3}\right].$$