Uma Antena Linear Centralmente Alimentada

Vamos considerar uma antena bem simples, consistindo de duas barras condutoras idênticas, retas e finas, ambas alinhadas ao longo de um eixo que vamos chamar de eixo z, mas com uma pequena separação entre elas, desprezível para nossa finalidade, onde o sinal elétrico é inserido. Vamos adotar a origem do sistema de coordenadas exatamente no centro dessa pequena separação entre as barras. Seja $d/2$ o comprimento de cada uma das barras condutoras. A ideia é supormos que a densidade de corrente elétrica estabelecida ao longo da antena seja simétrica com relação à origem de coordenadas e dependente senoidalmente de z e harmonicamente de t. Assim, vamos tomar a densidade de corrente complexa como

$$\mathbf{J}_c(\mathbf{r}) = \hat{\mathbf{z}} I \delta(x) \delta(y) \operatorname{sen}\left(\frac{kd}{2} - k|z|\right), \text{ para } |z| < \frac{d}{2},$$

sendo nula para $|z| \geqslant d/2$ e

$$k = \frac{\omega}{c}.$$

As funções de Dirac indicam que a antena é estreita sobre o eixo z. Na zona de radiação,

$$\begin{aligned} \mathbf{A}_c^{\mathrm{rad}}(\mathbf{r}) &= \frac{\exp(ikr)}{rc} \int_V d^3r' \mathbf{J}_c(\mathbf{r}') \exp(-ik\hat{\mathbf{r}} \cdot \mathbf{r}') \\ &= \frac{\exp(ikr)}{rc} \int_V d^3r' \hat{\mathbf{z}} I \delta(x') \delta(y') \operatorname{sen}\left(\frac{kd}{2} - k|z'|\right) \exp(-ik\hat{\mathbf{r}} \cdot \mathbf{r}'). \end{aligned}$$

Integrando em x' e y' fornece

$$\mathbf{A}_c^{\mathrm{rad}}(\mathbf{r}) = \hat{\mathbf{z}} \frac{I \exp(ikr)}{rc} \int_{-d/2}^{d/2} dz' \operatorname{sen}\left(\frac{kd}{2} - k|z'|\right) \exp(-ikz' \cos\theta),$$

onde θ é o ângulo entre o eixo z e o vetor posição do ponto de observação, $\mathbf{r}$. Calculemos a integral:

$$\begin{aligned} \int_{-d/2}^{+d/2} dz' \operatorname{sen}\left(\frac{kd}{2} - k|z'|\right) \exp(-ikz' \cos\theta) &= \int_{-d/2}^{0} dz' \operatorname{sen}\left(\frac{kd}{2} + kz'\right) \exp(-ikz' \cos\theta) \\ &+ \int_{0}^{d/2} dz' \operatorname{sen}\left(\frac{kd}{2} - kz'\right) \exp(-ikz' \cos\theta), \end{aligned}$$

que, com a mudança de variável

$$z' \to -z'$$

na primeira integral do membro direito, resulta em

$$\begin{aligned} \int_{-d/2}^{+d/2} dz' \operatorname{sen}\left(\frac{kd}{2} - k|z'|\right) \exp(-ikz' \cos\theta) &= \int_{0}^{d/2} dz' \operatorname{sen}\left(\frac{kd}{2} - kz'\right) \exp(ikz' \cos\theta) \\ &+ \int_{0}^{d/2} dz' \operatorname{sen}\left(\frac{kd}{2} - kz'\right) \exp(-ikz' \cos\theta) \end{aligned}$$

e, portanto,

$$\int_{-d/2}^{+d/2} dz'\text{sen}\left(\frac{kd}{2} - k\left|z'\right|\right)\exp\left(-ikz'\cos\theta\right) = 2\int_{0}^{d/2} dz'\text{sen}\left(\frac{kd}{2} - kz'\right)\cos\left(kz'\cos\theta\right).$$

Mas,

$$\begin{aligned}\text{sen}\left(\frac{kd}{2} - kz'\right)\cos\left(kz'\cos\theta\right) &= \frac{1}{2}\text{sen}\left(\frac{kd}{2} - kz' + kz'\cos\theta\right) + \frac{1}{2}\text{sen}\left(\frac{kd}{2} - kz' - kz'\cos\theta\right) \\ &= -\frac{1}{2}\text{sen}\left[k\left(1+\cos\theta\right)z' - \frac{kd}{2}\right] - \frac{1}{2}\text{sen}\left[k\left(1-\cos\theta\right)z' - \frac{kd}{2}\right]\end{aligned}$$

e, assim,

$$\begin{aligned}\int_{-d/2}^{+d/2} dz'\text{sen}\left(\frac{kd}{2} - k\left|z'\right|\right)\exp\left(-ikz'\cos\theta\right) &= -\int_{0}^{d/2} dz'\text{sen}\left[k\left(1+\cos\theta\right)z' - \frac{kd}{2}\right] \\ &- \int_{0}^{d/2} dz'\text{sen}\left[k\left(1-\cos\theta\right)z' - \frac{kd}{2}\right] \\ &= \frac{\cos\left[k\left(1+\cos\theta\right)\frac{d}{2} - \frac{kd}{2}\right] - \cos\left(\frac{kd}{2}\right)}{k\left(1+\cos\theta\right)} \\ &+ \frac{\cos\left[k\left(1-\cos\theta\right)\frac{d}{2} - \frac{kd}{2}\right] - \cos\left(\frac{kd}{2}\right)}{k\left(1-\cos\theta\right)}.\end{aligned}$$

Simplificando, obtemos

$$\begin{aligned}\int_{-d/2}^{+d/2} dz'\text{sen}\left(\frac{kd}{2} - k\left|z'\right|\right)\exp\left(-ikz'\cos\theta\right) &= \frac{1}{k}\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right] \\ &\times \left[\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta}\right] \\ &= \frac{2}{k}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\text{sen}^2\theta}\right].\end{aligned}$$

Logo,

$$\mathbf{A}_c^{\text{rad}}\left(\mathbf{r}\right) = \hat{\mathbf{z}}\frac{2I\exp\left(ikr\right)}{rck}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\text{sen}^2\theta}\right].$$

Calculemos o campo indução magnética de radiação:

$$\begin{aligned}\mathbf{B}_c\left(\mathbf{r}\right) &= \boldsymbol{\nabla}\times\mathbf{A}_c^{\text{rad}}\left(\mathbf{r}\right) \\ &= \boldsymbol{\nabla}\times\left\{\hat{\mathbf{z}}\frac{2I\exp\left(ikr\right)}{rck}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\text{sen}^2\theta}\right]\right\} \\ &= -\hat{\mathbf{z}}\times\boldsymbol{\nabla}\left\{\frac{2I\exp\left(ikr\right)}{rck}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\text{sen}^2\theta}\right]\right\}.\end{aligned}$$

Como

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\varphi}}}{r\mathrm{sen}\theta}\frac{\partial}{\partial \varphi},$$

segue

$$\begin{aligned}
\mathbf{B}_c(\mathbf{r}) &\approx -\hat{\mathbf{z}} \times \hat{\mathbf{r}}\frac{\partial}{\partial r}\left\{\frac{2I\exp(ikr)}{rck}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\mathrm{sen}^2\theta}\right]\right\} \\
&\approx -\hat{\mathbf{z}} \times \hat{\mathbf{r}}ik\frac{2I\exp(ikr)}{rck}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\mathrm{sen}^2\theta}\right]
\end{aligned}$$

e, como

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\mathrm{sen}\theta\cos\varphi + \hat{\mathbf{y}}\mathrm{sen}\theta\mathrm{sen}\varphi + \hat{\mathbf{z}}\cos\theta,$$

então

$$\begin{aligned}
\hat{\mathbf{z}} \times \hat{\mathbf{r}} &= \hat{\mathbf{z}} \times \hat{\mathbf{x}}\mathrm{sen}\theta\cos\varphi + \hat{\mathbf{z}} \times \hat{\mathbf{y}}\mathrm{sen}\theta\mathrm{sen}\varphi + \hat{\mathbf{z}} \times \hat{\mathbf{z}}\cos\theta \\
&= \hat{\mathbf{y}}\mathrm{sen}\theta\cos\varphi - \hat{\mathbf{x}}\mathrm{sen}\theta\mathrm{sen}\varphi \\
&= \mathrm{sen}\theta\left(\hat{\mathbf{y}}\cos\varphi - \hat{\mathbf{x}}\mathrm{sen}\varphi\right) \\
&= \mathrm{sen}\theta\hat{\boldsymbol{\varphi}}.
\end{aligned}$$

Portanto,

$$\mathbf{B}_c^{\mathrm{rad}}(\mathbf{r}) = -\hat{\boldsymbol{\varphi}}i\frac{2I\exp(ikr)}{rc}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right].$$

E o campo elétrico de radiação fica

$$\begin{aligned}
\mathbf{E}_c(\mathbf{r}) &= \frac{i}{k}\boldsymbol{\nabla} \times \mathbf{B}_c^{\mathrm{rad}}(\mathbf{r}) \\
&= \frac{i}{k}\boldsymbol{\nabla} \times \left\{-\hat{\boldsymbol{\varphi}}i\frac{2I\exp(ikr)}{rc}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right]\right\}.
\end{aligned}$$

Como

$$\begin{aligned}
\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{r}) &= \frac{\hat{\mathbf{r}}}{r\mathrm{sen}\theta}\left[\frac{\partial}{\partial\theta}\left(\mathrm{sen}\theta F_\varphi\right) - \frac{\partial F_\theta}{\partial\varphi}\right] + \hat{\boldsymbol{\theta}}\left[\frac{1}{r\mathrm{sen}\theta}\frac{\partial F_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}\left(rF_\varphi\right)\right] \\
&+ \frac{\hat{\boldsymbol{\varphi}}}{r}\left[\frac{\partial}{\partial r}\left(rF_\theta\right) - \frac{\partial F_r}{\partial\varphi}\right],
\end{aligned}$$

para um campo vetorial $\mathbf{F}(\mathbf{r})$ arbitrário, obtemos

$$\begin{aligned}
\mathbf{E}_c(\mathbf{r}) &= \frac{i}{k}\boldsymbol{\nabla} \times \left\{-\hat{\boldsymbol{\varphi}}i\frac{2I\exp(ikr)}{rc}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right]\right\}. \\
&= -\frac{i}{k}\hat{\mathbf{r}}i\frac{2I\exp(ikr)}{r^2c\mathrm{sen}\theta}\frac{\partial}{\partial\theta}\left[\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)\right] \\
&+ \frac{i}{k}\hat{\boldsymbol{\theta}}\frac{1}{r}i\frac{2Iik\exp(ikr)}{c}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right) - \cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right],
\end{aligned}$$

isto é,

$$\mathbf{E}_c^{\mathrm{rad}}(\mathbf{r}) = -\frac{i}{k}\hat{\boldsymbol{\theta}}\frac{2Ik\exp(ikr)}{rc}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right].$$

A média temporal do vetor de Poynting de radiação é dada por

$$\begin{aligned}\left\langle \mathbf{S}^{\mathrm{rad}}\right\rangle &= \frac{c}{8\pi}\mathrm{Re}\left\{\mathbf{E}_c^{\mathrm{rad}}(\mathbf{r})\times\left[\mathbf{B}_c^{\mathrm{rad}}(\mathbf{r})\right]^*\right\} \\ &= \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\varphi}}\frac{I^2}{2\pi r^2 c}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right]^2.\end{aligned}$$

Logo,

$$\left\langle \mathbf{S}^{\mathrm{rad}}\right\rangle = \hat{\mathbf{r}}\frac{I^2}{2\pi r^2 c}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right]^2$$

e a distribuição angular da média temporal da potência irradiada por unidade de área é dada por

$$\begin{aligned}\frac{dP}{d\Omega} &= \hat{\mathbf{r}}\cdot\left\langle \mathbf{S}^{\mathrm{rad}}\right\rangle r^2 \\ &= \frac{I^2}{2\pi c}\left[\frac{\cos\left(\frac{kd}{2}\cos\theta\right)-\cos\left(\frac{kd}{2}\right)}{\mathrm{sen}\theta}\right]^2.\end{aligned}$$